On the dimension growth of groups

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Honolulu, March 3, 2012

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For example if Γ is $\mathbb{Z}$ (or the square lattice $\mathbb{Z}^n$), then $k(1) = 2$.
Color even vertices in white, odd vertices in black.
The growth rate of $k(\lambda)$ is a q.i. invariant.

The upper bound

Proposition. The dimension growth of a finitely generated group G does not exceed its volume growth.

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Proof Let f be the volume growth function. We consider a graph with vertices elements of G where every two vertices at distance $\leq \lambda$ are joined by an edge. Then the valency of this graph is $\leq f(\lambda)$. The graph has chromatic number $\leq f(\lambda) + 1$.

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Corollary. The dimension growth of any finitely generated group is at most exponential.

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Hence Gromov random groups containing expanders have exponential asymptotic dimension growth. This is the only known example.

Distortion of subgroups and dimension growth

Observation. Suppose that for a group Γ $k_\Gamma(\lambda) = k$ for some λ. Suppose that Γ (L, C)-embeds into G. Then $k_G(\frac{\lambda - C}{L}) \geq k$.

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Theorem.(Panov, Moore) Let $\Gamma = \mathbb{Z}^\infty$ with l_1-metric. Then $k_\Gamma(2) = \infty$.

Proof. Every finite subset M of $\mathbb{N}$ corresponds to a vector $v(M)$ from $\mathbb{Z}^\infty$ with coordinates 0, 1 in the natural way. Choose any $k \geq 1$. Let $P_k(\mathbb{N})$ denote the set of all k-element subsets of $\mathbb{N}$. Every finite coloring of $\mathbb{Z}^\infty$ induces a finite coloring of $P_k(\mathbb{N})$. By Ramsey there exists a subset $M \subseteq \mathbb{N}$ of size $2k$ such that all k-element subsets of M have the same color. Therefore we can find subsets $T_1, T_2, \ldots, T_k$ of size k from M such that the symmetric distance between T_i and T_{i+1} is 2, $i = 1, \ldots, k-1$, and T_1, T_k are disjoint. Then the vectors $v(T_1), \ldots, v(T_k)$ from $\mathbb{Z}^\infty$ form a monochromatic 2-path of diameter $\geq 2k$.

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Idea of the proof. Extend the coloring of $\mathbb{Z}^n$ to coloring of $\mathbb{R}^n$ and use the fact that the covering dimension of $\mathbb{R}^n$ is n.

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Idea of the proof. $\mathbb{Z}^{\exp n}$ embeds into $\mathbb{Z} \wr G$ with q.i. constants $(O(n), 1)$. Hence $k_G(\lambda) \geq k_{\mathbb{Z}^{\exp n}}(\lambda / O(n))$. Take $\lambda = O(n^2)$. We conclude by the theorem above that $k_G(O(n^2)) \geq \exp O(n)$.

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Remark. This is the biggest known dimension growth function of an amenable group.

The lower bounds of the dimension growth of the R. Thompson group and its subgroups.

Theorem.(D+S, follows from Arzhantseva+Guba+S) The group F contains a $(n,1)$-distorted copy of $\mathbb{Z}^{2^n}$ for every n. Hence the dimension growth of F is at least $\exp\sqrt{n}$.

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Theorem. (D+S) There exists an elementary amenable subgroup B of F with $k_B(n) \geq \exp\sqrt{n}$.

An open problem

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If "yes", then the asymptotic dimension growth is exponential.
We do not know the answer for $\lambda = 2, \alpha = 1$. We also do not know whether $k_{\mathbb{Z}^n}(\lambda)$ is bounded for every λ as a function of n.

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Problem. What is the actual dimension growth of the iterated wreath product B_s?