The Tarski numbers of groups

Mark Sapir

With Mikhail Ershov and Gili Golan

The Tarski number

A group G admits a *paradoxical decomposition* if there exist positive integers m and n, disjoint subsets $P_1, \ldots, P_m$, $Q_1, \ldots, Q_n$ of G and elements $g_1, \ldots, g_m$, $h_1, \ldots, h_n$ of G such that

$$G = \bigcup_{i=1}^{m} g_i P_i = \bigcup_{j=1}^{n} h_j Q_j.$$

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The minimal possible value of $m + n$ in a paradoxical decomposition of G is the *Tarski number* of G and denoted by $\mathcal{T}(G)$.

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$$P_1 \supseteq G \setminus gP_2 \supseteq g(P_1 \cup P_2 \cup Q_1 \cup Q_2) \setminus gP_2 = g(P_1 \cup Q_1 \cup Q_2).$$

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Hence $P_1 \supset g^m P_1 \supset g^{m+1}(Q_1 \cup Q_2)$ for every $m \geq 0$.

$$P_2 \supseteq G \setminus g^{-1} P_1 \supseteq g(P_1 \cup P_2 \cup Q_1 \cup Q_2) \setminus g^{-1} P_1 = g^{-1}(P_2 \cup Q_1 \cup Q_2).$$

Hence $P_2 \supset g^m P_2 \supset g^{m-1}(Q_1 \cup Q_2)$ for every $m \leq 0$.

Known facts II

Ceccherini-Silberstein, Grigorchuk, de la Harpe:

- The Tarski number of any torsion group is at least 6.
- The Tarski number of any non-cyclic free Burnside group of odd exponent ≥ 665 is between 6 and 14.

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It is possible to show that the Tarski number of every group from Fin_k is at least $2k+4$.

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It is not known what t is exactly. The estimate: 10^{10^8}. The case $t = 4$ is Farb's problem.

Graph-theoretic formulation

Let G be a group, S_1, S_2 be finite subsets of G. Consider the Cayley graph $Cay(G, \{S_1, S_2\})$, color edges in two colors. A subgraph is called an evenly colored 2-graph if every vertex has two children of different colors, and at most one parent.

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Tarski numbers of extensions

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(a) Suppose that H has finite index in G. Then

$$\mathcal{T}(H) - 2 \leq [G : H](\mathcal{T}(G) - 2).$$

(b) Let $\mathcal{V}$ be a variety of groups where all groups are amenable and relatively free groups are right orderable. Then there exists a function $f : \mathbb{N} \to \mathbb{N}$ (depending only on $\mathcal{V}$) with the following property: if H is normal in G and $G/H \in \mathcal{V}$, then $\mathcal{T}(H) \leq f(\mathcal{T}(G))$.

(c) Assume that H is normal and amenable. Then $\mathcal{T}(G/H) = \mathcal{T}(G)$.

(d) Assume that $G = H \times K$ for some K. Then $\min\{\mathcal{T}(H), \mathcal{T}(K)\} \leq 2(\mathcal{T}(G) - 1)^2$.

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Suppose that G has a paradoxical decomposition with translating sets S_1, S_2 and assume that $1 \in S_1 \cap S_2$. Let $S = S_1 \cup S_2$. Then let $S_i' = TS_iT^{-1} \cap H$. Then H has a paradoxical decomposition with translating sets S_1', S_2'. Therefore, $\mathcal{T}(H) \leq |S_1'| + |S_2'|$.

Proof continued

To show this, consider an evenly colored 2-subgraph Γ of the Cayley graph $Cay(G, \{S_1, S_2\})$, and identify vertices with the same H-components. The new graph has vertex set H.

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We prove that it satisfies the conditions of Hall's lemma, so it contains an evenly colored 2-subgraph. The edges are labeled by elements of S_i', so we get a subgraph of $Cay(H, \{S_1', S_2'\})$. Thus the Tarski number of H is at most $|S_1'| + |S_2'|$.

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We estimate $|S_i'| = |TS_iT^{-1} \cap H| \leq |T|(|S_i| - 1) + 1$ (here we use the fact that S_i contains 1.)

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We estimate $|S_i'| = |TS_iT^{-1} \cap H| \leq |T|(|S_i| - 1) + 1$ (here we use the fact that S_i contains 1.) Hence $\mathcal{T}(H) \leq |S_1'|+|S_2'| \leq |T|(|S_1|+|S_2|-2)+2 = [G:H](\mathcal{T}(G)-2)+2$, and we are done.

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We use Neumann-Neumann construction and the fact that free metabelian groups are left orderable.

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LEMMA. Suppose that a group G is generated by a set $T = \{a, b, c\}$ of 3 non-identity elements, and suppose that $|\partial^+_T A| \geq |A|$ for every finite subset $A \subseteq G$. Then G admits a paradoxical decomposition with both translating sets of size 3, and therefore $\mathcal{T}(G) \leq 6$.

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Proof. Let $S_1 = \{1, a, b\}, S_2 = \{1, b, c\}$. By Hall's lemma, Γ has a spanning 2-subgraph which can be evenly colored because every 2-element subset of $\{1, a, b, c\}$ can be ordered so that the first element is in S_1, the second is in S_2.

Tarski number 6, continued

THEOREM. Let G be any 3-generated group with $\beta_1(G) \geq 3/2$ where $\beta_1(G)$ is the first L^2-Betti number of G. Then $\mathcal{T}(G) \leq 6$. In particular, if G is torsion, then $\mathcal{T}(G) = 6$.

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LEMMA. Let G be a finitely generated group, S a finite generating subset of G, and let $k = 2\beta_1(G) - |S| + 1$. Then for any finite $A \subseteq G$ we have $|\partial_S^+ A| \geq k|A|$.

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PROOF. First find a subforest F with $\sum_{v\in A} \deg_F(v) \geq (2\beta_1(G) + 2)|A|$ (Lyons).

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Then remove edges with negative labels. That gives $|\partial^+(A)| \geq (2\beta_1(G) + 2 - |S|)|A| - |A|$.

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$|\partial^+(A)| \geq (2\beta_1(G) + 2 - |S|)|A| - |A|$.

By Osin's theorem, there are torsion 3-generated groups with $\beta_1(G) > 3/2$, all these groups have Tarski numbers 6.

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PROBLEM 3. Suppose that $\beta_1(G) > 0$. Is it true that $\mathcal{T}(G) \leq 6$?
Peterson and Thom: if G is torsion-free, $\beta_1 > 0$, and Atiyah's conjecture holds, then $\mathcal{T}(G) = 4$.