On the conjugacy growth functions of groups

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The definition

Definition. Let $G = \langle X \rangle$ be a group generated by a finite set X. For every n let $g_c(n)$ be the number of conjugacy classes of G intersecting the ball of radius n in G. The function $g_c(n)$ will be called the conjugacy growth function of G.

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It does not affect the result in the cases of hyperbolic, relatively hyperbolic or CAT(0)-spaces and groups.

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There are no examples of finitely presented groups with exponential growth function and subexponential conjugacy growth function.

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One can also use a result of Kropholler characterizing finitely generated solvable groups which do not have sections which are wreath products of a cyclic group with $\mathbb{Z}$ and results of Osin about the uniform growth of solvable groups.

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Problem 1. Compute the conjugacy growth of Grigorchuk groups.

Baumslag-Solitar

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Proof. It is easy to see that for numbers $k \neq l$ not divisible by n, the elements a^k, a^l are not conjugate in $BS(1,n)$.

The group $S_\infty \rtimes \mathbb{Z}$

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Problem 2. Find more precise estimates for the conjugacy growth functions of finitely generated nilpotent groups.

Diagram groups. Definition. I

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Diagram group. Definition. II

Let X be an alphabet. For every $x \in X$ we define the *trivial diagram* $\varepsilon(x)$ which is just an edge labeled by x. The top and bottom paths of $\varepsilon(x)$ are equal to $\varepsilon(x)$, $\iota(\varepsilon(x))$ and $\tau(\varepsilon(x))$ are the initial and terminal vertices of the edge. If u and v are words in X, a *cell* $(u \to v)$ is a planar graph consisting of two directed labeled paths, the top path labeled by u and the bottom path labeled by v, connecting the same points $\iota(u \to v)$ and $\tau(u \to v)$.

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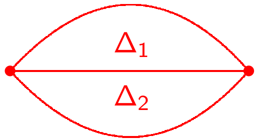

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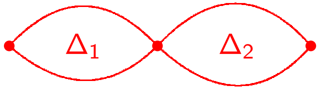

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Diagram groups. Definition. III

Definition. A diagram over a collection of cells P is any planar graph obtained from the trivial diagrams and cells of P by the operations of addition, multiplication and inversion. If the top path of a diagram Δ is labeled by a word u and the bottom path is labeled by a word v, then we call Δ a (u, v)-diagram over P.

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Let $P = \{c_1, c_2, \ldots\}$ be a collection of cells. The diagram group $DG(P, u)$ corresponding to the collection of cells P and a word u consists of all reduced (u, u)-diagrams obtained from these cells and trivial diagrams by using the three operations mentioned above. The product $\Delta_1 \Delta_2$ of two diagrams Δ_1 and Δ_2 is the reduced diagram obtained by removing all dipoles from $\Delta_1 \circ \Delta_2$.

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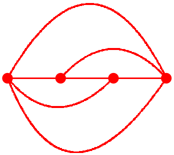
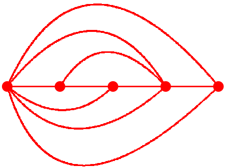


x_0



x_1

Diagram metric. Burillo's property.

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$1 \equiv 2$, $3 \to 1$ are true. $1 \to 3$ is still unknown.

Conjugacy growth of diagram groups with property B

Theorem. Every finitely generated diagram group with B containing the wreath product $\mathbb{Z} \wr \mathbb{Z}$ (in particular, the R.Thompson group F) has exponential conjugacy growth function.

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(ii) Suppose that two absolutely reduced diagrams A and B have canonical decompositions $A_1 + \cdots + A_m$ and $B_1 + \cdots + B_n$ (where A_i is a (u_i, u_i)-diagram, B_j is a (v_j, v_j)-diagram). Suppose further that A and B are conjugate. Then $m = n$, and A_i is conjugate to B_i, that is $A_i = \Gamma_i^{-1} B_i \Gamma_i$ for some (v_i, u_i)-diagram Γ_i, $i = 1, \ldots, m$.

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(iii) If two simple diagrams A, B are conjugate then they have the same number of cells.

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$$\varepsilon(a) + \pi^{n_0} + \cdots + \pi^{n_k} + \varepsilon(c).$$

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There is a diagram Γ such that $A(n_0, ..., n_k) = \Gamma^{-1}\Delta(n_0, ..., n_k)\Gamma \in DG(P, ac)$.

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By the theorem, the number of pairwise non-conjugate diagrams $A(n_0, ..., n_k)$ with $n_0 + ... + n_k = n$ is 2^n.

The rigidity

Theorem 1. Suppose that for some collection of cells Q and some word u we have $DG(Q, u) \geq \mathbb{Z} \wr \mathbb{Z}$.

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Theorem 1. Suppose that for some collection of cells Q and some word u we have $DG(Q, u) \geq \mathbb{Z} \wr \mathbb{Z}$. Then there exists a *natural* embedding Ψ of $\mathbb{Z} \wr \mathbb{Z}$ into $DG(Q, u)$. It is *induced* by a diagram Γ, and a map ψ that takes letters a, b, c to words $\psi(a), \psi(b), \psi(c)$ over the alphabet of Q, and each of the three cells $x \to y$ of P to a non-trivial $(\psi(x), \psi(y))$-diagram $\psi(x \to y)$ over Q.

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The end of the proof.

Consider the natural embedding Ψ of $\mathbb{Z} \wr \mathbb{Z}$ into $DG(Q, u)$. The diagrams $\Psi(A(n_0, ..., n_k)$ pairwise are not conjugate.

A conjecture

Conjecture 4. Suppose that G acts on a simplicial tree non-trivially and faithfully. Then the conjugacy growth function of G is exponential provided the growth function of G is exponential.

A theorem

Theorem 2. Let G be the HNN extension of a group H with associated subgroups A, B such that $AB \cup BA \neq H$. Then the conjugacy growth function of G is exponential.

The proof

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Let the presentation of G consist of all relations of H plus the conjugacy relations $ut = tv$ of the HNN-extension (here $u \in A, v \in B$).

The proof, continued

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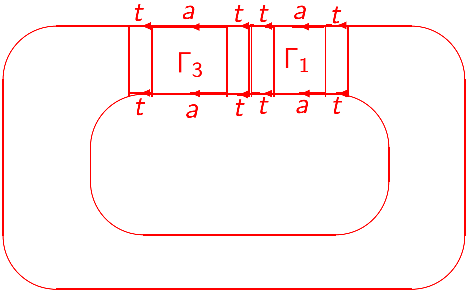

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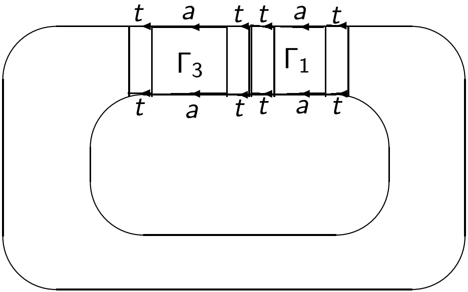

The t-bands give a correspondence between the a-edges on the boundary. The condition $a \notin AB \cup BA$ implies that the correspondence is a cyclic shift.

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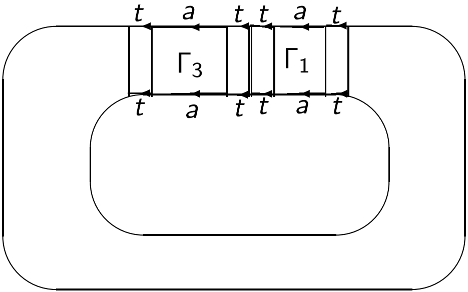

The t-bands give a correspondence between the a-edges on the boundary. The condition $a \notin AB \cup BA$ implies that the correspondence is a cyclic shift. Hence the conjugacy growth function of G is exponential.