Polynomial maps over fields and residually finite groups

Alexander Borisov and Mark Sapir[1]

[1] Inventiones Math., 2005

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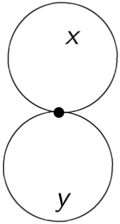

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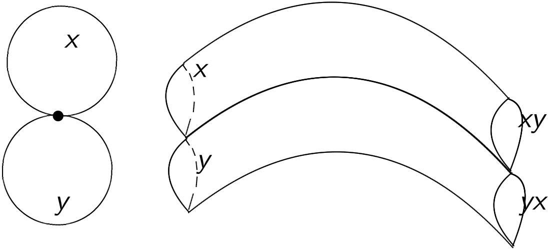

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- (Feighn-Handel) If G is free then $\mathrm{HNN}_{\phi}(G)$ is *coherent* i.e. every f.g. subgroup is f.p.
- (Geoghegan-Mihalik-S.-Wise) If G is free then $\mathrm{HNN}_{\phi}(G)$ is *Hopfian* i.e. every surjective endomorphism is injective.

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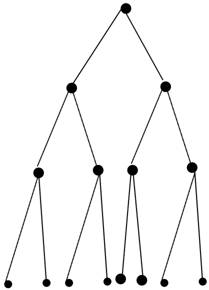

An automorphism f must fix the root and so it fixes the levels of the tree.

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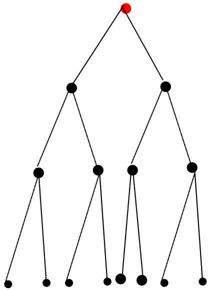

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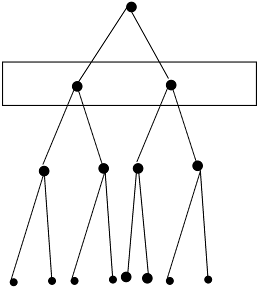

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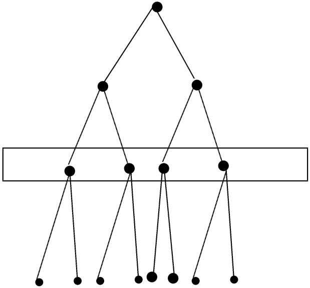

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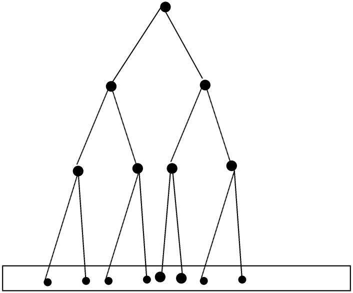

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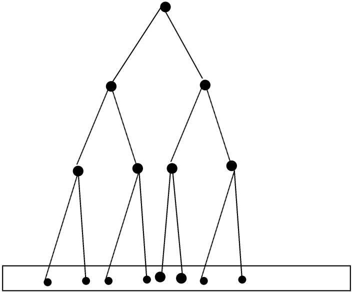

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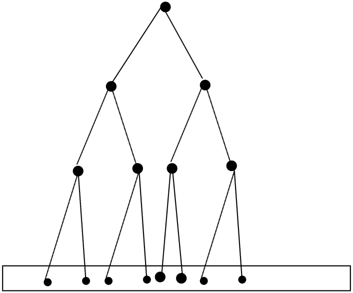

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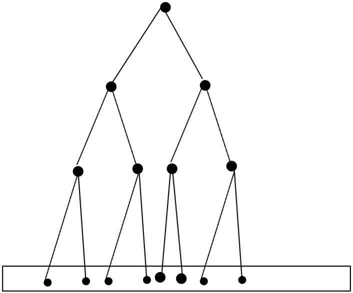

Conversely every finitely generated residually finite group acts faithfully on a locally finite rooted tree.

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Problem. (Moldavanskii, Kapovich, Wise) Are ascending HNN extensions of free groups residually finite?
These three problems are related.

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So we have a new presentation of the same group.

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Note that b_{-1} appears only once in $b_1 b_0^{-1} b_1 b_0^{-1} b_{-1}^{-1} = 1$.

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So we can replace b_{-1} by $b_1 b_0^{-1} b_1 b_0^{-1}$, remove this generator, and get a new presentation of the same group.

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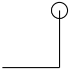

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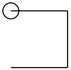

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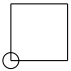

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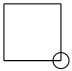

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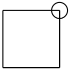

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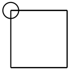

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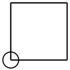

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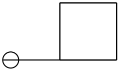

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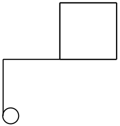

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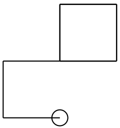
Indexes of b's are coordinates of the vertical steps of the walk.



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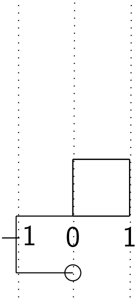

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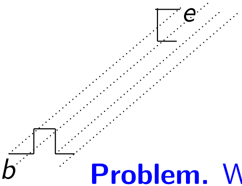

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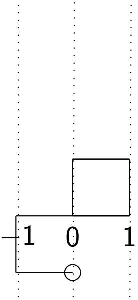

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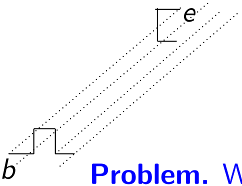

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Dunfield and Thurston proved recently that this probability is strictly between 0 and 1.

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So $(\bar{x}, \bar{y})$ is a periodic point of the map

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on the "space" $V \times V$.

A problem from algebraic geometry over groups.

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Key observation. The converse statement is also true (the number of generators and the choice of ϕ do not matter).

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Example continued

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Thus the point (A, B) is periodic in $\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$ with period 6.

Example continued. Dynamics of polynomial maps over local fields

Replace 5 by 25, 125, etc. It turned out that (A, B) is periodic in $SL_2(\mathbb{Z}/25\mathbb{Z})$ with period 30, in $\mathrm{SL}_2(\mathbb{Z}/125\mathbb{Z})$ with period 150, etc.

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Therefore our group $\langle a, b, t \mid tat^{-1} = ab, tbt^{-1} = ba\rangle$ is residually finite.

Polynomial maps over finite fields

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Thus our problem is reduced to the following:

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Let $G = \langle a_1, ..., a_k, t \mid t a_i t^{-1} = u_i, i = 1, ..., k \rangle$.Consider the ring of matrices $M_2(\mathbb{Z})$.

The map $\psi \colon M_2(\mathbb{Z})^k \to M_2(\mathbb{Z})^k$ is given by

$$\vec{x} \mapsto (w_1(\vec{x}), ..., w_k(\vec{x})).$$

It can be considered as a polynomial map $A^{4k} \to A^{4k}$. The group we consider will be $\mathrm{PGL}_2(.)$. Here "." is any finite field of, say, characteristic p.

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Thus our problem is reduced to the following:

Problem. Let P be a polynomial map $A^n \to A^n$ with integer coefficients. Show that the set of periodic points of P is Zariski dense.

Counting periodic points. Quasi-fixed points

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Theorem (Borisov, Sapir). Every ascending HNN extension of a free group is residually finite.

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Conjecture: No.

Proof

We denote by I_Q the ideal in $\overline{\mathbb{F}_q}[x_1, ..., x_n]$ generated by the polynomials $f_i(x_1, ..., x_n) - x_i^Q$, for $i = 1, 2, ..., n$.

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Let us fix some polynomial D with the coefficients in a finite extension of $\mathbb{F}_q$ such that it vanishes on W but not on V.

Proof continued

Step 4. There exists a positive integer K such that for all quasi-fixed points $(a_1, \ldots, a_n) \in W$ with big enough Q we get

$$R = (D(f_1^{(n)}(x_1, \ldots, x_n), \ldots, f_n^{(n)}(x_1, \ldots, x_n)))^K \equiv 0 (\mod I_Q^{(a_1, \ldots, a_n)})$$

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This implies that $R \in I_Q$.

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This means that there exist polynomials $u_1, ...u_n$ such that

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Step 7. We look how the monomials cancel in the equation (1) and get a contradiction.