On the dimension growth of groups

Alexander Dranishnikov and Mark Sapir

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The covering dimension

Definition (approximate). The dimension of a space X is at most n if for every $\epsilon > 0$ there exists an (open) coloring in at most $n + 1$ colors such that every monochromatic path has diameter at most ϵ.

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The growth rate of $k(\lambda)$ is a q.i. invariant.

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$\mathbb{Z} \wr \mathbb{Z}$, the Grigorchuk group, the R. Thompson group F, etc. have infinite asymptotic dimension and the question about dimension growth is natural for these groups.

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$D(\lambda)$ is linear, and $k(\lambda)$ is constant, we get the Assouad-Nagata dimension.
We do not know any finitely generated group where more than exponential control is needed.

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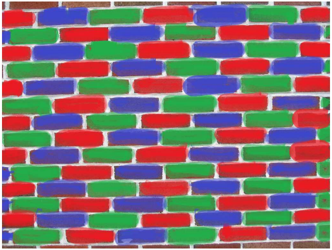

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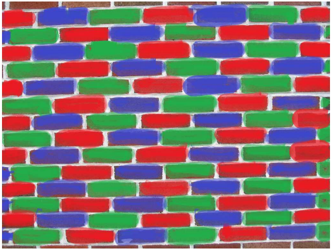

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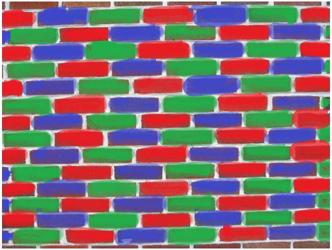

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Problem. What is the smallest size of a brick (as a function in n)?
What if we color in a different way (not by bricks)?

Connection with quasi-isometry and coarse embeddings

A map of metric spaces $\phi : X \to Y$ is called a *coarse embedding* if there are strictly monotone tending to infinity functions $\rho_1, \rho_2 : \mathbb{R}_+ \to \mathbb{R}_+$ and a number $r > 0$ such that

$$\rho_1(d_X(x, x')) \leq d_Y(\phi(x), \phi(x')) \leq \rho_2(d_X(x, x'))$$

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Let $\phi : X \to Y$ be a coarse embedding with functions ρ_1, ρ_2: Then

$$(\lambda, D)\text{-dim}(Y) \geq (\rho_2^{-1}(\lambda), \rho_1^{-1}(D))\text{-dim}(X).$$

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Proof Let f be the volume growth function. We consider a graph with vertices elements of G where every two vertices at distance $\leq \lambda$ are joined by an edge. Then the valency of this graph is $\leq f(\lambda)$. The graph has chromatic number $\leq f(\lambda) + 1$.

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Corollary. The dimension growth of any finitely generated group is at most exponential (with any control since the size of every cluster is 1, does not depend on λ).

Connection with functional analysis

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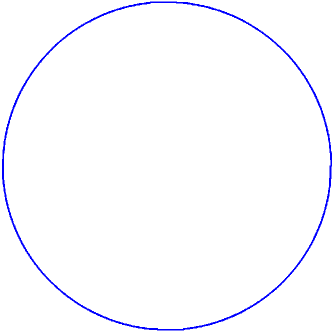

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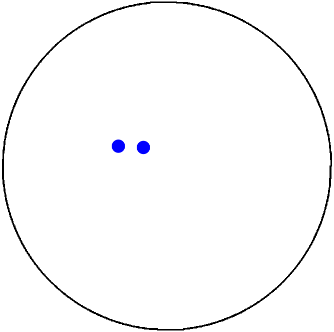

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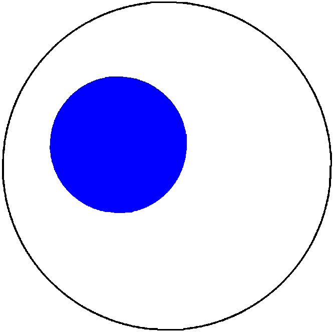

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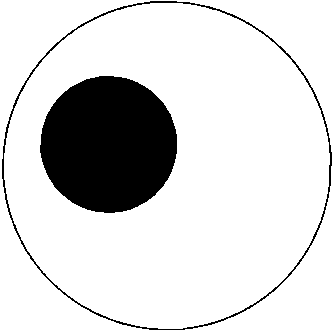

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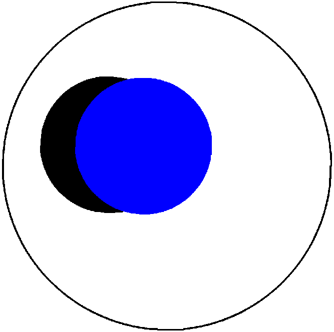

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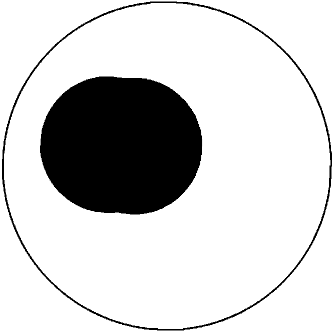

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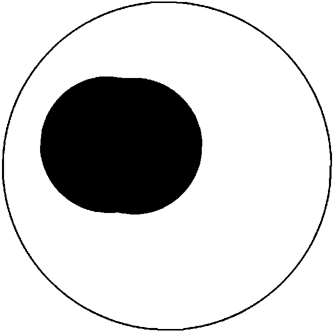

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Hence Gromov random groups containing expanders have exponential asymptotic dimension growth.

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Then the dimension growth of G is exponential.
Proof. Let $V_G = \cup_{i=1}^{k+1} U_i$ be a coloring of the vertices of G in $k+1$ colors such that all λ-clusters U_i^j have diameters at most d. Take $r > d + \lambda$ and consider the graph $G_r = (V_r, E_r)$. Let $W_i^j = U_i^j \cap G_r$. We have $\cup W_i^j$ equal to the set V_r of all vertices of G_r. Note that $N_{\lambda/2}(W_i^j)$ has at least $(1+\varepsilon)^{\lambda/2}$ elements. Since different λ-clusters of the same color are λ-disjoint, we have that the sum of $|N_{\lambda/2}(W_i^j)|$ is at most $|V_r|(k+1)$. On the other hand, that sum is at least $(1+\varepsilon)^{\lambda/2}$ times the sum of cardinalities $|W_i^j|$, i.e. at least $|V_r|(1+\varepsilon)^{\lambda/2}$. Hence $k+1 \geq (1+\varepsilon)^{\lambda/2}$.

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Proof. Every finite subset M of $\mathbb{N}$ corresponds to a vector $v(M)$ from $\mathbb{Z}^\infty$ with coordinates 0, 1 in the natural way. Choose any $k \geq 1$. Let $P_k(\mathbb{N})$ denote the set of all k-element subsets of $\mathbb{N}$. Every finite coloring of $\mathbb{Z}^\infty$ induces a finite coloring of $P_k(\mathbb{N})$. By Ramsey there exists a subset $M \subseteq \mathbb{N}$ of size $2k$ such that all k-element subsets of M have the same color. Therefore we can find subsets $T_1, T_2, \ldots, T_k$ of size k from M such that the symmetric distance between T_i and T_{i+1} is 2, $i = 1, \ldots, k-1$, and T_1, T_k are disjoint. Then the vectors $v(T_1), \ldots, v(T_k)$ from $\mathbb{Z}^\infty$ form a monochromatic 2-path of diameter $\geq 2k$.

The dimension growth of $\mathbb{Z}^n$.

Let G be the binary cube $\{0,1\}^n$ with the ℓ_1-metric. Then for every $r > 0$, such that $\varepsilon = \frac{n}{r+1} - 2 > 0$, G satisfies property $(P_r(\epsilon))$.

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[**The controlled 4-dimension of a binary *n*-cube**] The binary n-cube $\{0,1\}^n$, $n > 64$, cannot be colored by n colors such that each 4-cluster of every color has diameter less than $\leq \sqrt{n}/4$, i.e. $(4, \sqrt{n}/4)$-dim $(\mathbb{Z}^n) = n$ for $n > 64$.

The main open problem.

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If "yes", then the asymptotic dimension growth of F is exponential.
We do not know the answer for $\lambda = 2, \alpha = 1$. We also do not know whether $k_{\mathbb{Z}^n}(\lambda)$ is bounded for every λ as a function of n.

Connection with the game of Hex

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Theorem. There is always a winner in the game of Hex. Hence if we color $\mathbb{Z}^n$ with ℓ_∞-metric in n colors there is always arbitrary long monochromatic paths. Thus 1-dim$(\mathbb{Z}^n, \ell_\infty) = n$ for every n.

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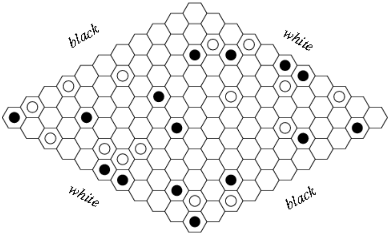
Remark. The game of Hex on the plane corresponds to the hexagonal tessellation of the plane and the graph metric on its dual graph. As we know from percolation theory (Smirnov), hexagonal lattice is much easier than square lattice.



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Hence the dimension growth of F with some exponential control is exponential.

What is the dimension growth of F? Is super-exponential control required?

Upper bounds. Connection with the Kolmogorov-Ostrand dimension

We say that Kolmogorov-Ostrand dimension of X is $\leq n$ if for every $m \geq 0$ there exists a coloring of X in $m + n$ colors (every point may be colored in many colors) such that the diameters of all λ-clusters are uniformly bounded.

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For Assouad-Nagata dimension it was proved by Brodskiy, Dydak, Levin, and Mitra.

Proof. Suppose $KO-dim(X) = n_1, KO-dim(Y) = n_2$. Consider colorings of X and Y in n_1+n_2+m colors (as required by the definition). Then color (x, y) in color i if both x and y has color i. This gives a required coloring of $X \times Y$.

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Problem. What is the dimension growth of $\mathbb{Z} \wr \mathbb{Z}$?