Polynomial maps over fields and residually finite groups

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Lecture 4. Applications.

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- Residually finite,
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- Coherent (that is all finitely generated subgroups are finitely presented).

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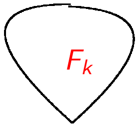

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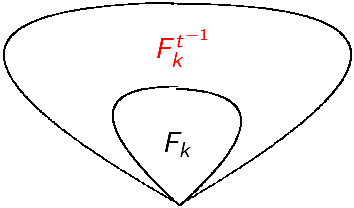

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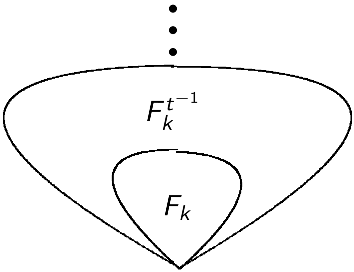

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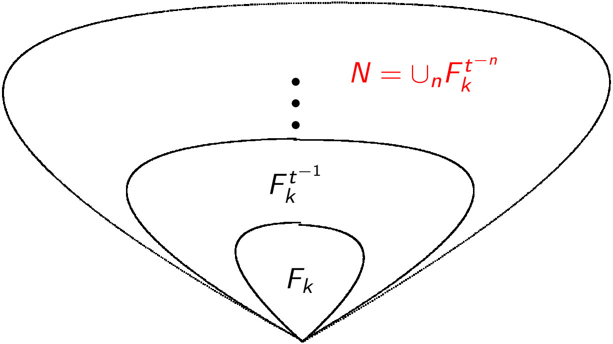

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- in order to find such a pair of matrices with the additional property that (A, B) generate a free subgroup, we use a result of Breuillard and Gelander the matrices A, B are found in the p-adic completion of $SL_2(\mathcal{O})$.

What to do next? Non-residually finite hyperbolic groups.

Consider double HNN extensions of free groups. For example,
$H = \langle x, y, t, s \mid x^t = xy, y^t = yx, x^s = [x, y], y^s = [x^2, y^2] \rangle$.

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HW2. Find a finite simple non-Abelian homomorphic image of H.

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$$\begin{aligned} H(k,i,w) = &\langle a_1, ..., a_k, t \mid ta_1t^{-1} = a_2, ..., ta_{i-1}t^{-1} = a_i, \\ &ta_it^{-1} = w, twt^{-1} = a_{i+1}, \\ &ta_{i+1}t^{-1} = t_{i+2}, ..., ta_{k-1}t^{-1} = a_k\rangle. \end{aligned}$$

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Question 2. Is every group $H(k,i,w)$ residually finite?