Polynomial maps over fields and residually finite groups

Mark Sapir

August, 2009, Bath, UK
The lectures are based on the following three papers:
The lectures are based on the following three papers: Alexander Borisov, Mark Sapir, Polynomial maps over finite fields and residual finiteness of mapping tori of group endomorphisms. Invent. Math. 160 (2005), no. 2, 341–356.
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Iva Kozáková, Mark Sapir, Almost all one-relator groups with at least three generators are residually finite. preprint, arXiv math0809.4693, 2008.
Lecture 1. Around 1-related groups.
Residually finite groups

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**Examples.** $\mathbb{Z}, F_k$, linear groups are residually finite. $\mathbb{Q}$, infinite simple groups, free Burnside groups of sufficiently large exponents are not residually finite. Groups acting faithfully on rooted locally finite trees are residually finite.
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Conversely every finitely generated residually finite group acts faithfully on a locally finite rooted tree.
Linear groups

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(A. Malcev, 1940) Every finitely generated linear group is residually finite. Moreover, it is virtually residually (finite $p$-)group for all but finitely many primes $p$. Note that a linear group itself may not be residually (finite $p$-)group for any $p$. Example: $\text{SL}_3(\mathbb{Z})$ by the Margulis normal subgroup theorem.
Problem. Is every hyperbolic group residually finite?
Problems.

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**Problem.** When is a one-relator group $\langle X \mid R = 1 \rangle$ residually finite?
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**Example 2.** $BS(1, 2) \langle a, t | tat^{-1} = a^2 \rangle$ is metabelian, and linear, so it is residually finite.
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**Model 1.** Uniform distribution on words of length $\leq n$. 
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**Model 3.** Uniform distribution on 1-related groups given by cyclically reduced relators of length $\leq n$ (up to isomorphism)
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These models are equivalent. 3 $\equiv$ 1: I. Kapovich-Schupp.
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Hyperbolic groups and 1-related groups

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Fact 3 and a result of P. Neumann imply Fact 2.
Proof of Fact 3

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The standard Reidermeister-Schreier shows that $\bar{H}$ has $(g - 1)(n - m) + l$ generators $s, s_{j,i}, j = 2, ..., g, m \leq i \leq n$, and $nr$ relators not involving $s$. 
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So $\bar{H}$ is $\langle s \rangle * K$ where $K$ has $(g - 1)(n - m)$ generators and $nr$ relators. For large enough $n$, then $\#\text{generators} - \#\text{relators of } K$ is $\geq 1$. So $K$ maps onto $\mathbb{Z}$, and $\bar{H}$ maps onto $F_2$. Q.E.D.
Theorem (Lackenby) For every large group $G$ and every $g \in G$ there exists $n$ such that $G/\langle \langle g^n \rangle \rangle$ is large.
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**Proof (Olshanskii-Osin)** Same as Baumslag-Pride.
Lackenby, Olshanskii-Osin and the Burnside problem

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**Application.** There exists an infinite finitely generated group that is:

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- every finite section is solvable; every nilpotent finite section is Abelian.
Example (Magnus procedure).

Consider the group \( \langle a, b \mid aba^{-1}b^{-1}aba^{-1}b^{-1}a^{-1}b^{-1}a = 1 \rangle \).
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Replace \(a^i ba^{-i}\) by \(b_i\). The index \(i\) is called the *Magnus a-index* of that letter.
Example (Magnus procedure).

\[ \langle a, b_{-1}, b_0, b_1 \mid b_1 b_0^{-1} b_1 b_0^{-1} b_{-1}^{-1} = 1, a^{-1} b_0 a = b_{-1}, a^{-1} b_1 a = b_0 \rangle. \]
So we have a new presentation of the same group.
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Note that \( b_{-1} \) appears only once in \( b_1 b_0^{-1} b_1 b_0^{-1} b_{-1}^{-1} = 1. \)
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So we can replace $b_{-1}$ by $b_1 b_0^{-1} b_1 b_0^{-1}$, remove this generator, and get a new presentation of the same group.
Example (Magnus procedure).

\[ \langle a, b_0, b_1 \mid a^{-1}b_0a = b_1b_0^{-1}b_1b_0^{-1}, \quad a^{-1}b_1a = b_0 \rangle. \]  
This is clearly an ascending HNN extension of the free group \( \langle b_0, b_1 \rangle \).
Ascending HNN extensions

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![Diagram of HNN extension](attachment:diagram.png)
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Facts about ascending HNN extensions

- Every element in an ascending HNN extension of $G$ can be represented in the form $t^{-k}gt^\ell$ for some $k, \ell \in \mathbb{Z}$ and $g \in G$. 
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- (Feighn-Handel) If $G$ is free then $\text{HNN}_\phi(G)$ is coherent i.e. every f.g. subgroup is f.p.
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- (Wise-S.) An ascending HNN extension of a residually finite group can be non-residually finite (example - Grigorcuk’s group and its Lysenok extension).
Walks in $\mathbb{Z}^2$

Consider the word $aba^{-1} \cdot b^{-1} \cdot aba^{-1} \cdot b^{-1} \cdot a^{-1} b^{-1} a$ and the corresponding walk on the plane:
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\[ 
\begin{array}{c}
\circ \\
\end{array}
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Magnus indexes of $b$’s are coordinates of the vertical steps of the walk.
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Magnus indexes of $b$’s are coordinates of the vertical steps of the walk.
Walks in $\mathbb{Z}^2$

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In general:

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Dunfield and Thurston proved recently that this probability is strictly between 0 and 1.
Ken Brown’s results

Let $G = \langle x_1, \ldots, x_k \mid R = 1 \rangle$ be a 1-relator group.
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- If $k = 2$ and one of the two support lines of $w$ that is parallel to $\vec{OM}$ intersects $w$ in a single vertex or a single edge,
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- If \( k = 2 \) and one of the two support lines of \( w \) that is parallel to \( \overrightarrow{OM} \) intersects \( w \) in a single vertex or a single edge, then \( G \) is an ascending HNN extension of a free group.
- If \( k > 2 \) then \( G \) is never an ascending HNN extension of a free group.