You could have noticed that we started a completely new topic, "Rings". I assume that you have an algebra book and can read there or in your notes all the definitions needed to solve these problems.

0 (*) Let $R$ be an (associative unitary) ring. Suppose that $a \in R$ is a nilpotent element. Prove that $1 - a$ is a unit.

1. (*) Let $Q$ be the quaternion ring (see your notes, by the way, it was introduced by Hamilton).
   a) For every $q = a + bi + cj + dk \in Q$, let the norm $|q|$ be defined as $\sqrt{a^2 + b^2 + c^2 + d^2}$ (here $a, b, c, d$ are real numbers). Prove that for every $q_1, q_2 \in Q$, we have $|q_1| |q_2| = |q_1q_2|$. 
   b) Let $Q_Z$ be the subset of $Q$ consisting of all quaternions with integer coefficients. Show that $Q_Z$ is a subring of $Q$.
   c) Show that an element $q$ of $Q_Z$ is a unit of $Q_Z$ if and only if $|q| = 1$.
   d) Find all $x \in Q$ satisfying $x^2 = i$?

2. (*) Let $R$ be a ring. The smallest non-zero natural number $p$ such that $p \cdot 1 = 1 + 1 + ... + 1$ ($p$ times) is equal to 0 in $R$ (if such a $p$ exists), is called the characteristic of $R$. If such a $p$ does not exist, we say that $R$ has characteristic 0. Prove that the characteristic of a division ring is a prime number or 0.

3. (*) Let $G$ be a group, $H < G$ be a finite subgroup of $G$, $|H| = n$. Let $R$ be a ring and $R[G]$ be the group ring of $G$ with coefficients from $R$. Suppose that $n \cdot 1 = 1 + 1 + ... + 1$ ($n$ times) is a unit in $R$. Prove that the element $e = (n \cdot 1)^{-1} \sum_{h \in H} h$ of $R[G]$ is an idempotent.

4. (*) Show that every abelian group $A$ with operation + becomes an associative (but not unitary) ring if we define the product by $a \ast b = 0$ for all $a, b \in A$.

5. (*) Let $A$ be an abelian group. Consider the set of all endomorphisms $End(A)$ of $A$ (i.e. all homomorphisms from $A$ to $A$). If $f, g \in End(A)$ then we define $f + g$ as the map $A \to A$ which takes every $a$ to $f(a) + g(a)$. We also define $f \cdot g$ as the composition of $f$ and $g$. Show that $End(A)$ is a ring.

6. (*) a) Find all idempotents and nilpotent elements in the ring $\mathbb{Z}/12\mathbb{Z}$.
   b) Find all idempotents and nilpotent elements in the ring of 2 by 2 matrices $M_2(\mathbb{Z}/2\mathbb{Z})$. 

1
c) Let $F_1, F_2$ be division rings. Find all idempotents and nilpotent elements in the direct product $F_1 \times F_2$.

7. Let $R$ be a non-unitary associative ring. Consider the set $R^1$ consisting of all formal sums $n + r$ where $n \in \mathbb{Z}, r \in R$. Define operations on $R^1$ by setting $(n + r) + (n' + r') = (n + n') + (r + r')$ and $(n + r)(n' + r') = nn' + (nr' + n'r + rr')$ where $nr'$, for example, is $r' + r' + \ldots + r'$ ($n$ times). Show that $R^1$ is a unitary associative ring containing $R$ as a (non-unitary) subring. Thus one can make every ring into a unitary ring.

8. (***) Let $\mathbb{Z}_p$ be the ring of all $p$-adic numbers (see the notes or the book).
   
   a) (***) Prove that if $p = 5$ there exists an element $x$ of $\mathbb{Z}_5$ satisfying $x^2 = -1$.
   
   b) (*) Prove that if $p = 3$ then there is no $x \in \mathbb{Z}_3$ satisfying $x^2 = -1$. 