The complexity of a problem is indicated by the number of ∗ in front of it.

**Notation.** Let $G$ act on $X$, $g \in G$, $x \in X$. Then the result of applying the map corresponding to $g$ to $x$ is denoted $g \circ x$.

0. a) (∗) The group $\mathbb{Z}/4\mathbb{Z}$ (numbers modulo 4) acts on itself by addition on the right: $a \circ b = b + a$. Write down permutations of the set $\{0, 1, 2, 3\}$ corresponding to elements of $\mathbb{Z}/4\mathbb{Z}$ (for example, 0 corresponds to the trivial permutation).

b) (∗) The Quaternion group $Q_8$ acts on itself by conjugation: $a \circ b = a^{-1}ba$. Write down the permutations of the 8-element set $Q_8$ which correspond to all elements of $Q_8$.

c) (**) The group $S_4$ of all permutations of $\{1, 2, 3, 4\}$ acts on the set $Y$ of all 2-element subsets of the set $\{1, 2, 3, 4\}$ in the natural way: if $a \in S_4$, $\{i, j\}$ is a 2-element subset of $\{1, 2, 3, 4\}$ then $a \circ \{i, j\} = \{a(i), a(j)\}$. Right down the permutation of the 6-element set $Y$ corresponding to $(1, 2, 3, 4) \in S_4$. Find all the orbits of this action and stabilizers of each element of $Y$.

1. Consider two transformations of the plane: $T$ is the rotation through angle $\pi/2$, $P$ is the reflection about the $x$-axis. Consider the subgroup $H$ generated by $T$ and $P$.

   a) (∗) Prove that $H$ has 8 elements, 4 rotations and 4 reflections.

   b) (∗) Is this group isomorphic to $Q_8$?

   c) (∗) Describe the orbit of the point $(0, 5)$ under this group. What is the stabilizer of this point in the group?

   d) (∗) Find a fundamental domain of this group of isometries.

2. (**) Consider the subgroup of the group of isometries of the plane generated by the rotation $T_\alpha$ through angle $\alpha$, and the reflection about the $x$-axis. Depending on $\alpha$, how many elements does this group have? Describe geometrically the orbit of the point $(1, 0)$ under this group.

3. (****) Let $G(\alpha)$ be the group of isometries of the plane generated by the rotation $T_\alpha$ and the translation by vector $(1, 0)$. For which $\alpha$ the group $G(\alpha)$ has a fundamental domain?

4. (∗) Let $S_4$ be the group of permutations of a 4-element set, let $H$ be the subgroup of $S_4$ generated by the permutation $(1, 2, 3, 4)$ Find all left and right cosets of this subgroup.

**Definition.** Let $G$ be a group generated by a finite set $X$. The Cayley graph $\text{Cay}(G, X)$ of $G$ with respect to $X$ is the labeled graph where vertices are elements of $G$, and edges labeled by $x \in X$ connect $g$ with $gx$ for every $g \in G$. Notice that $\text{Cay}(G, X)$ is always connected.
We assign to every edge of this graph length 1. Then the set of vertices of this graph can be considered a metric space if the distance between two vertices is defined as the length of the shortest path between these vertices. Notice that the metric “forgets” about the labels of edges.

5. (a) (**) Consider the action of $G$ on $\text{Cay}(G, X)$: $h \circ g = h^{-1}g$. Show that $G$ acts on $\text{Cay}(G, X)$ by isometries.

(b) (**) Give an example of $G$ for which $G$ does not coincide with the group of all isometries of the metric space $\text{Cay}(G, X)$.

(c) (****) Find a modification $C$ of the metric space $\text{Cay}(G, X)$ such that $G = \text{Iso}(C)$. 