Counting Number Fields by Discriminant and Point Counting on Varieties

Eric Larson and Larry Rolen

Harvard University and Emory University

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Question

What are the asymptotics of this function?

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- Using "higher composition laws", Bhargava has studied the cases $G = S_4, S_5$.

Our Work

Theorem (L-R 2011)

We have

$$N(n, A_n, X) \ll_n X^{\frac{n^2-2}{4(n-1)}} \cdot \log(X)^{2n+1}.$$

Progress in the D_5 case

Theorem (L-R 2011)

To any quintic number field K with Galois group D_5, there corresponds a triple (A, B, C) with $A, B \in \mathcal{O}_{\mathbb{Q}[\sqrt{5}]}$ and $C \in \mathbb{Z}$, such that

$$\mathrm{Nm}_{\mathbb{Q}}^{\mathbb{Q}[\sqrt{5}]}\left(B^2 - 4 \cdot \bar{A} \cdot A^2\right) = 5 \cdot C^2$$

and which satisfies the following under any archimedean valuation:

$$|A| \ll D_K^{\frac{1}{4}}, \quad |B| \ll D_K^{\frac{3}{8}}, \quad \text{and} \quad |C| \ll D_K^{\frac{3}{4}}.$$

Conversely, the triple (A, B, C) uniquely determines K.

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Remark

We also provide numerical evidence that $N(5, D_5, X) \ll X^{\frac{2}{3}}$.

General Method of Point Counting

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- Let $R = \mathbb{Z}[x_1, \ldots, x_n]^G/(s_1)$ where $s_1 = x_1 + \ldots + x_n$.
- Every pair (K, α) gives a $\mathbb{Z}$-point of $\operatorname{Spec} R$ with bounded coordinates.

The Case of D_5

- Recall that it suffices to understand bounded $\mathbb{Z}$-points of

$$\operatorname{Spec} \mathbb{Q}[x_1, x_2, x_3, x_4, x_5]^{D_5}/(x_1 + x_2 + x_3 + x_4 + x_5).$$

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- Let $V_j := \sum_{i=1}^{5} \zeta^{ij} x_i$.
- Now define:

$$\begin{aligned} A &= V_2 \cdot V_3 \\ B &= V_1 \cdot V_2^2 + V_3^2 \cdot V_4 \\ C &= \frac{1}{\sqrt{5}} \cdot (V_1 \cdot V_2^2 - V_3^2 \cdot V_4) \cdot (V_2 \cdot V_4^2 - V_1^2 \cdot V_3). \end{aligned}$$

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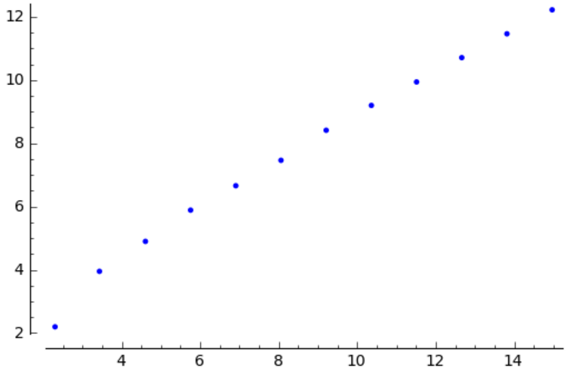
- The generators of D_5 act by $V_j \mapsto V_{5-j}$ and $V_j \mapsto \zeta^j V_j$.
- This easily gives the norm equation in the theorem.
- The fact that (A, B, C) uniquely determines K can be shown using the expressions for V_i explicitly.

Numerical Data and Remarks



- This log plot and a regression analysis give strong evidence that $N(5, D_5, X) \ll X^{\frac{2}{3}}$. The data goes up to $X = 3162277$.

The Case of A_n

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- For large n, Ellenberg and Venkatesh obtain:

$$N(n, A_n, X) \ll (X \cdot B_n)^{\exp(C \log \sqrt{n})}.$$

Our Result

Theorem (L-R 2011)

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- This bound is about $X^{\frac{1}{4}}$ better than Schmidt's bound.
- This is the best-known bound for $6 \leq n \leq 84393$.
- By a conjecture of Malle, we expect that $N(n, A_N, X) \overset{?}{\sim} X^{\frac{1}{2}}$.

Sketch of Proof

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- The case when n is even is easier; we will use covering spaces to prove it when n is odd.

The Case when n is Even

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- Each of these curves is the zero locus of a polynomial

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- In particular, these curves are all *geometrically irreducible*.

Pila's Bound and the Proof when n is even

Theorem (Pila 1996)

Let Γ be a geometrically irreducible plane curve of degree $d \geq 2$ and let S be a square of side $N \geq 2$ in the plane with sides parallel to the coordinate axes.

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- This immediately implies the result when n is even.

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Definition

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Definition

We say that c is a critical value of a polynomial f if $c = f(d)$ for some d with $f'(d) = 0$.

Two Lemmas on Critical Values

Lemma

Fix a finite set of points $S \subset \mathbb{C}$ and an integer d. Then there are finitely many equivalence classes of polynomials of degree d whose set of critical values is contained in S.

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Let n be an integer. For any monic polynomial $p(z) \in \mathbb{C}[z]$ of degree $n-1$, there are only finitely many values of $(a_2, a_3, \cdots, a_{n-1}) \in \mathbb{C}^{n-2}$ such that $p(z)$ is the discriminant of the polynomial

$$q(t) = t^n + a_2 t^{n-2} + \cdots + a_{n-1} t - z.$$

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- The proofs follow using basic theory of covering spaces.

Proof of Theorem when n is even

- Our curve is geometrically reducible iff $p(y) = \mathrm{disc}(t^n + s_2 t^{n-2} + \cdots \pm s_{n-1} t - y)$ is a perfect square.

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- The coefficients of $p(y)$ are regular functions in $s_2, s_3, \ldots, s_{n-1}$ and the induced map $\mathbb{A}^{n-2} \to \mathbb{A}^{n-1}$ is a finite map by the previous lemma.

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- The locus of $(b_1, b_2, \ldots, b_{n-1}) \in \mathbb{A}^{n-1}$ such that $t^{n-1} + b_1 t^{n-2} + \cdots + b_{n-1}$ is a perfect square is a Zariski-closed set of dimension $\frac{n-1}{2}$.

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- The proof now follows in a similar way as when n is even.

Conclusion

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Theorem

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$$\mathrm{Nm}_{\mathbb{Q}}^{\mathbb{Q}[\sqrt{5}]}\left(B^2 - 4 \cdot \bar{A} \cdot A^2\right) = 5 \cdot C^2 \tag{1}$$

and which satisfies the following under any archimedean valuation:

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Conversely, the triple (A, B, C) uniquely determines K.