Maass Forms and Quantum Modular Forms

Larry Rolen

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June 26, 2013
Modular Forms

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**Definition**

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2. $\left| (cz + d)^{-k} f \left( \frac{az+b}{cz+d} \right) \right| \ll e^{C \cdot \Im z}$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$. 
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- If \( k = 0 \), we call \( f \) a modular function.
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**Remarks**

- If $k = 0$, we call $f$ a modular function.
- We can also define modular forms of half-integral weight.
Congruence Subgroups

We are mainly interested in modular forms on groups like:

$$
\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}
$$
Any modular form of level $N$ has a Fourier expansion

$$f(z) = \sum_{n \gg -\infty} a_n q^n,$$

where $q := e^{2\pi i z}$. 

Fourier Expansions
Examples

1. The $j$-invariant is a modular function of level 1:

$$j(z) = q^{-1} + 744 + 196884q + \ldots$$

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$$\Delta(z) := q \prod_{n \geq 1} (1 - q^n)^{24}.$$  

3. The weight $\frac{1}{2}$ Jacobi theta function

$$\theta(z) := \sum_{n \in \mathbb{Z}} q^{n^2}.$$
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Singular Moduli

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- Zagier defined “traces of singular moduli”, which he proved are often coefficients of modular forms.

- We consider integrality for the polynomials arising from non-holomorphic functions.
For a positive-definite quadratic form $Q = ax^2 + bxy + cy^2$, let
Traces of Singular Moduli

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$$\tau_Q := \frac{-b + \sqrt{b^2 - 4ac}}{2a} \in \mathbb{H}.$$

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Definition

Let $Q_d$ be the set of positive definite binary quadratic forms of discriminant $d$. For a modular function $F$, define the trace:

$$\text{Tr}_d(F) := \sum_{Q \in Q_d/\Gamma} w_Q^{-1} F(\tau_Q).$$
An Example of Zagier’s Theory

**Theorem (Zagier)**

Let

\[ J(z) := j(z) - 744 \]

and

\[ g(z) := \theta_1(z) \frac{E_4(4z)}{\eta(4z)^6} = \sum B(d)q^n \]

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\[ \text{Tr}_{-d} (J(z)) = -B(d). \]
Another Example; \( K := \partial \left( \frac{E_4 E_6}{\Delta} \right) \)

Define \( H_d(K; x) := \prod_{Q \in Q_d/\Gamma} (x - K(\tau_Q)) \).
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- $H_{-23}(K; x) = x^3 - 23261998 x^2 - \frac{3945271661}{23} x - 7693330369871$. 

Remark: It appears that the third symmetric function is always an integer.
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Traces for Negative Weight Forms

- Recall the **Maass raising operator**, which raises the weight of a Maass form by 2:

  \[ R_k := 2i \frac{\partial}{\partial z} + ky^{-1}. \]

- For \( f \) of negative weight, \( \partial f \) is the **iterated** raising to weight 0.
Theorem 1 (Griffin-R 2012)

Let \( f(z) \in M_k^! \), \( 0 > k \in 2\mathbb{Z} \) have integral principal part. Denote the \( n^{th} \) symmetric function in the singular moduli of discriminant \( d \) for \( \partial f \) by \( S_f(n; d) \). Let

\[
B(n, k) := \begin{cases} 
\frac{-nk}{4} & \text{if } nk \in 4\mathbb{Z} \\
\frac{1}{4}(-nk + 2k - 2) & \text{otherwise.}
\end{cases}
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Then we have that

\[
d^{B(n,k)} \cdot S_f(n; d) \in \mathbb{Z}.
\]
Corollary

For any $f(z) \in M_{-2}$ with integral principal part, we have that

$$S_f(3; d) \in \mathbb{Z}.$$
Corollary

For any $f(z) \in M^1_{-2}$ with integral principal part, we have that

$$S_f(3; d) \in \mathbb{Z}.$$ 

Remark

This theorem is sharp.
Sketch of Proof

- Use Newton’s identities to reduce to sums of powers.
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- Unfortunately, powers of Maass forms are usually not finite sums of Maass forms.
Theorem (Griffin-R 2012)

Let $F$ be a product of “raises” of modular forms. Then there are modular forms $g_j \in M_{k-2j}^!$ such that

$$F = \sum_{j=0}^\infty R_j g_j.$$
The Spectral Decomposition

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The proof gives an explicit algorithm for computing the forms $g_j$. 
Sketch of Proof (cont).

- Work of Duke and Jenkins allows us to study integrality of traces for $\partial f$ when $f$ is a negative weight modular form.
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- Bounding denominators on each piece gives a naïve bound.
Two Intervening Problems

- Obstruction 1: Certain weights in the decomposition give the wrong denominators.
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- **Obstruction 2:** The coefficients $c_{i,j}$ in the previous theorem also introduce artificial denominators.
Two Intervening Problems

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- **We prove a vanishing condition on which forms in the decomposition actually appear.**

- **Obstruction 2:** The coefficients $c_{i,j}$ in the previous theorem also introduce artificial denominators.

- **We show that they cancel using the action of the Hecke algebra on Poincaré series.**

  Q.E.D.
Let $f \in M_k^1, \ g \in M_\ell^1, \ n \in \mathbb{N}$. The $n^{th}$ Rankin-Cohen bracket is

$$\left[ f, g \right]_{n}^{(k, \ell)} := \sum_{r+s=n} (-1)^r (n+k-1)s (n+\ell-1)r f(r) \cdot g(s).$$

This gives a map

$$\left[ \cdot, \cdot \right]^{(k)}_{n} : M_k^1 \otimes M_\ell^1 \rightarrow M_{k+\ell+2n}.$$
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Rankin-Cohen Brackets

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Obstruction 1: Vanishing lemma

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- In this case, we can expand in terms of Rankin-Cohen brackets.
- Using a calculation of Beyerl-James-Trentacoste-Xue, this reduces to a binomial sum identity, for $j$ odd

$$
\sum_{m=0}^{s} (-1)^{j+m} \cdot \frac{(m+r)(s)(m-r-1)}{(-r-2s+m+j-1)} = 0.
$$
Obstruction 2: Lining Up Principal Parts

- Raise the Zagier lifts of the pieces to the same weight and let:

\[ Z(\tau) := \sum_{t=0}^{\left\lfloor \frac{E+1}{2} \right\rfloor} (-1)^{M+t} R^{M+t} 3_1(g_{2t-1}) + \sum_{t=0}^{M} (-1)^{M+t} R^{M-t} 3_1(g_{2t}). \]
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- By comparison with \( F \), we observe that the holomorphic part \( Z^+ \) of \( Z \) has integral principal part.

- If all the coefficients of \( Z^+ \) are integral, then the \( c_{i,j} \)-denominators will cancel.
Maass-Poincaré Series

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- Thus, for any \( F(\tau) = \sum a(n)q^n \in M_{-2k}^! \) we can write
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- Thus, for any $F(\tau) = \sum a(n)q^n \in M^!_{-2k}$ we can write

$$F = \sum_{n<0} a(n)n^{1+2k}f_{-2k,1}|T(n).$$
Maass-Poincaré Series

- Maass-Poincaré series provide convenient bases.
- Thus, for any $F(\tau) = \sum a(n)q^n \in M^!_{-2k}$ we can write
  \[ F = \sum_{n<0} a(n)n^{1+2k}f_{-2k,1}T(n). \]
- The Zagier lift is equivariant with the Hecke action:
  \[ \mathcal{Z}_D(f|T(n)) = \mathcal{Z}_D(f)|T(n^2). \]
We construct a family of Hecke operators with “nice properties”.

**Corollary**

If \( f_{k,1} \mid H \) has integer coefficients, \( p \) is ordinary for all eigenforms in a basis of \( S_k \), and \( f_{k,1} \mid H \equiv 0 + O(q) \pmod{p^n} \), then
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**Corollary**

If $f_{k,1}|H$ has integer coefficients, $p$ is ordinary for all eigenforms in a basis of $S_k$, and $f_{k,1}|H \equiv 0 + O(q) \pmod{p^n}$, then

$$f_{k,1}|H \equiv 0 \pmod{p^n}.$$
A Tricky Question

Consider the integral

\[ \int_{\alpha}^{i\infty} \frac{\eta(2z)^2/\eta(z)}{(z - \alpha)^{3/2}} \, dz. \]

**Question**

*How does one evaluate it?*
What can we do?
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Corollary

We give exact values for all of these integrals as algebraic multiples of $\pi$ by specializing one formula.
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Functions on $\mathbb{Q}$ which are modular up to a “nice function”.
Quantum Modular Forms

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- Functions on $\mathbb{Q}$ which are modular up to a “nice function”.

- They have connections to: unimodal sequences, ranks, cranks, Dedekind sums, Eichler integrals, mock theta functions . . .
Defining Quantum Modular Forms

**Definition**

We say that a function $f : \mathbb{Q} \to \mathbb{C}$ is a *quantum modular form* if

$$f(x) - f|_{k\gamma}(x) = h_{\gamma}(x),$$

where $h_{\gamma}(x)$ is a “nice” function.
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A “Strange” Quantum Modular Form

A striking example of quantum modularity is given by the Kontsevich “strange function”:

\[ F(q) = \sum_{n=0}^{\infty} (1 - q)(1 - q^2) \cdots (1 - q^n) = \sum_{n=0}^{\infty} (q; q)_n. \]
A “Strange” Quantum Modular Form

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Remark

This function is strange as it is not defined on any open subset of \( \mathbb{C} \), but is well-defined at roots of unity.
Zagier’s Result

Theorem (Zagier)

$e^{\pi i x/12} F(e^{2\pi i x})$ is a wt. $3/2$ quantum modular form.
A New Quantum Modular Form

- We consider sums of tails of other eta-quotients.
A New Quantum Modular Form

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- We study the vector-valued form:

\[
H(q) = \begin{pmatrix}
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\theta_2 \\
\theta_3
\end{pmatrix} := \begin{pmatrix}
\frac{\eta(z)^2}{\eta(2z)} \\
\frac{\eta(z)^2}{\eta(z/2)} \\
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- We then associate finite versions \( \theta_{i,n} \) so that \( \theta_{i,n} \to \theta_i \).
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- We then associate finite versions \(\theta_{i,n}\) so that \(\theta_{i,n} \to \theta_i\).
- The corresponding “strange” function is \(\theta_i^S := \sum_{n=0}^{\infty} \theta_{i,n}\), which converges on some set of roots of unity.
Theorem 2 (R-Schneider 2012)

There are q-series $G_i$ also defined for $|q| < 1$ with

$$\theta_i^S(q^{-1}) = G_i(q).$$
A Vector-Valued Quantum Modular Form

Theorem 2 (R-Schneider 2012)

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  \[ \theta_i^S(q^{-1}) = G_i(q). \]

- We find $(\theta_1^S, \theta_2^S, \theta_3^S)^T$ is a wt. 3/2 quantum modular form.
Numerical Examples

- Our results give finite expressions for period integrals:
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Let \( \mathcal{I}(\alpha, x) := \int_{\alpha + x^{-1}}^{x \cdot i} \frac{\theta_1(z)}{(z - \alpha)^3} \, dz \).

\[ \mathcal{I}(1/5, 1/2) \sim -7.1250 + 18.0078i \]
\[ \mathcal{I}(1/5, 3/2) \sim 12.078 + 35.7274i \]
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\[ \mathcal{I}(1, 3/2) \sim 52.0472 + 25.685i \]
\[ \mathcal{I}(1, 5/2) \sim 76.4120 - 28.9837i \]
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<table>
<thead>
<tr>
<th>( k )</th>
<th>( \pi i(i + 1)\theta_1^5(\zeta_k) )</th>
<th>( I(1/k, 10^9) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>( \pi i(i + 1)(-2\zeta_3 + 3) \sim -7.1250 + 18.0078i )</td>
<td>( -7.1249 + 18.0078i )</td>
</tr>
<tr>
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<td>( \pi i(i + 1)(-2\zeta_5^3 - 2\zeta_5^2 - 8\zeta_5 + 3) \sim 12.078 + 35.7274i )</td>
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<td>( \pi i(i + 1)(6\zeta_7^4 - 2\zeta_7^2 - 10\zeta_7 + 7) \sim 52.0472 + 25.685i )</td>
<td>( 52.0474 + 25.685i )</td>
</tr>
<tr>
<td>9</td>
<td>( \pi i(i + 1)(8\zeta_9^4 - 16\zeta_9 + 3) \sim 76.4120 - 28.9837i )</td>
<td>( 76.4116 - 28.9836i )</td>
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Zagier’s Idea

- The proof comes from a “sum of tails” identity:

\[
\sum_{n=0}^{\infty} \left( \eta(24z) - q(1-q^{24})(1-q^{48})\cdots(1-q^{24n}) \right) = \eta(24z)D(q) + E(q)
\]

where \(E(q)\) is a “half-derivative” of \(\eta(24z)\). Thus, \(F(q)\) equals a half-derivative of \(\eta(24z)\) at roots of unity. Such a half-derivative is equal to an “Eichler integral”, but now the integral lives in \(\mathbb{H}\) and agrees at rationals.
Zagier’s Idea

- The proof comes from a “sum of tails” identity:

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\sum_{n=0}^{\infty} \left( \eta(24z) - q(1 - q^{24})(1 - q^{48}) \cdots (1 - q^{24n}) \right) = \eta(24z)D(q) + E(q)
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where \( E(q) \) is a “half-derivative” of \( \eta(24z) \).
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- Thus, \(F(q)\) equals a half-derivative of \(\eta(24z)\) at roots of unity.

- Such a half-derivative is equal to an “Eichler integral”, but now the integral lives in \(\mathbb{H}^-\) and agrees at rationals.
Sketch of the Proof

- The modularity of Eichler integrals comes from modularity of the original $\theta$-functions.
Sketch of the Proof

The modularity of Eichler integrals comes from modularity of the original $\theta$-functions.

Our strategy is as follows:

Strange function $\underset{\text{Sum of tails}}{\leftrightarrow}$ Half-Derivatives $\underset{\text{Reflection}}{\leftrightarrow}$ Eichler Integral
Let $F_9(z) := \eta(z)^2 / \eta(2z)$, and $F_{10}(z) := \eta(16z)^2 / \eta(8z)$.
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**Theorem (Andrews, Jimenez-Urroz, Ono)**

As formal power series, we have

\[
\sum_{n=0}^{\infty} (F_9(z) - F_{9,n}(z)) = 2F_9(z)E_1(z) + 2\sqrt{\theta}(F_9(z)),
\]

\[
\sum_{n=0}^{\infty} (F_{10}(z) - F_{10,n}(z)) = F_{10}(z)E_2(z) + \sqrt{\theta}(F_{10}(z)).
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Sums of Tails Identities

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\]

- Here $\sqrt{\theta} \sum a(n)q^n := \sum \sqrt{n}a(n)q^n$. 
For a weight $k$ cusp form $\sum a(n)q^n$, $k > 2$, the Eichler integral $\mathcal{E}_f$ is
Classical Eichler Integrals

For a weight \( k \) cusp form \( \sum a(n)q^n \), \( k > 2 \), the Eichler integral \( \mathcal{E}_f \) is

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\mathcal{E}_f := \sum n^{1-k} a(n)q^n.
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Classical Eichler Integrals

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Recall that $\mathcal{E}_f$ is modular up to a period polynomial:
Classical Eichler Integrals

- For a weight $k$ cusp form $\sum a(n)q^n$, $k > 2$, the Eichler integral $E_f$ is
  \[ E_f := \sum n^{1-k}a(n)q^n. \]

- Recall that $E_f$ is modular up to a period polynomial:
  \[ g(x) := c_k \int_0^\infty f(z)(z - x)^{k-2} \, dz. \]
If $k = 1/2$, the Eichler integral is a “half-derivative”.

A half-integral degree period polynomial (or the integral itself) is not well-defined. This can be fixed by defining an integral in the lower half plane which agrees with $\sqrt{\theta}(f)$ at rationals. The obstruction to modularity is not a polynomial, but it is still a $C^\infty$-function on $\mathbb{R}$. 

Larry Rolen
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Proof of the Theorem

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\[
H(z + 1) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & \zeta_{12} \\
0 & \zeta_{24} & 0
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Proof of the Theorem

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\[
H(-1/z) = \left(\frac{z}{i}\right)^{\frac{1}{2}} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} H(z).
\]
Proof of the Theorem (cont.)

- Extension of the strange functions to the upper half plane (after reflection) follows from power series manipulations, e.g.

\[
\theta_1^S(q^{-1}) = 2 \sum_{n=0}^{\infty} \frac{q^{2n+1}(q; q)_{2n}}{(1 + q^{2n+1})(-q; q)_{2n}}.
\]
The great anticipator of mathematics

Srinivasa Ramanujan (1887-1920)
“Death bed letter”

“Dear Hardy, I am extremely sorry for not writing you a single letter up to now. I discovered very interesting functions recently which I call “Mock” $\vartheta$-functions. Unlike the “False” $\vartheta$-functions (partially studied by Rogers), they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples.”

Ramanujan, January 12, 1920.
The first example

\[ f(q) = 1 + \frac{q}{(1 + q)^2} + \frac{q^4}{(1 + q)^2(1 + q^2)^2} + \cdots \]
Zwegers’ Work

In his Ph.D. thesis under Zagier ('02), Zwegers investigated:
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- “Lerch-type” series and Mordell integrals.

Ramanujan’s mock \( \vartheta \) functions are holomorphic parts of weight 1/2 harmonic Maass forms.
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“Theorem”

Ramanujan’s mock theta functions are holomorphic parts of weight $1/2$ harmonic Maass forms.
Notation. Throughout, let $z = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$. 
Defining Maass forms

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Hyperbolic Laplacian.

$$\Delta_k := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$
Harmonic Maass forms

“Definition”

A *harmonic Maass form* is any smooth function \( f \) on \( \mathbb{H} \) satisfying:

1. For all \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset \text{SL}_2(\mathbb{Z}) \), we have
   \[
   f(\gamma z) = (cz + d)^k f(z).
   \]
2. We have that \( \Delta^k f = 0 \).
Harmonic Maass forms

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A **harmonic Maass form** is any smooth function $f$ on $\mathbb{H}$ satisfying:

1. For all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset \text{SL}_2(\mathbb{Z})$ we have

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 f \left( \frac{az + b}{cz + d} \right) = (cz + d)^k f(z).
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Harmonic Maass forms

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HMFs have two parts

"Fundamental Lemma"

If $f \in H_{2-k}$ and $\Gamma(a, x)$ is the incomplete $\Gamma$-function, then

$$f(z) = \sum_{n \gg -\infty} c_f^+(n)q^n + \sum_{n<0} c_f^-(n)\Gamma(k - 1, 4\pi|n|y)q^n.$$  

Holomorphic part $f^+$  Nonholomorphic part $f^-$

Remark

The mock theta functions are examples of $f^+$. 
So many recent applications

- $q$-series and partitions
- Modular $L$-functions (e.g. BSD numbers)
- Eichler-Shimura theory
- Probability models
- Generalized Borcherds products
- Moonshine for affine Lie superalgebras and $M_{24}$
- Donaldson invariants
- Black holes
- ...
Is there more?

Ramanujan's last letter.

Asymptotics, near roots of unity, of "Eulerian modular forms".

Raises one question and conjectures the answer.

Gives one example supporting his conjectured answer.

Concludes with a list of his mock theta functions.
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Ramanujan’s question

Question (Ramanujan)

_Must Eulerian series with “similar asymptotics” be the sum of a modular form and a function which is \( O(1) \) at all roots of unity?_
Ramanujan’s answer

The answer is it is not necessarily so.

When it is not so I call the function
Mock $\vartheta$-function. I have not proved
rigorously that it is not necessarily so. But I have constructed a number
of examples in which it is not in-
conceivable to construct a $\vartheta$ func-
tion to cut out the singularities.
Ramanujan’s last words

“it is inconceivable to construct a $\vartheta$ function to cut out the singularities of a mock theta function...”

Srinivasa Ramanujan
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“...it has not been proved that any of Ramanujan’s mock theta functions really are mock theta functions according to his definition.”

Bruce Berndt (2012)
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Theorem 3 (Griffin-Ono-R 2013)

*Ramanujan’s examples satisfy his own definition.*
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Bruce Berndt (2012)

Theorem 3 (Griffin-Ono-R 2013)

Ramanujan’s examples satisfy his own definition. More precisely, a mock theta function and a modular form never cut out exactly the same singularities.
Sketch of proof: parallel weight

- Suppose a mock theta function $f$ of weight $k$ is cut out by a modular form $g$ of weight $k'$. 
Sketch of proof: parallel weight

- Suppose a mock theta function $f$ of weight $k$ is cut out by a modular form $g$ of weight $k'$.  

- By the Bruinier-Funke pairing, any HMF has a nonzero principal part at some cusp.
Sketch of proof: different weights

- We have that $c_f^-(n)$ are supported on finitely many square classes, so we can kill $f^-$ with quadratic twists.
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- If $f$ cut out $g$, then $\tilde{f}$ cuts out $\tilde{g}$ where $\tilde{g}$ is the result of twisting $g$.

- We ruled out the case $k = k'$. If $k \neq k'$, it is easy to show this cannot happen for two modular forms.
Here I have discussed results on:
Conclusion

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- Symmetric functions in singular moduli for nonholomorphic modular functions.
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- A new example of a quantum modular form.
Conclusion

Here I have discussed results on:

- Symmetric functions in singular moduli for nonholomorphic modular functions.
- A new example of a quantum modular form.
- Ramanujan’s original definition of a mock modular form.
Further results

I have also proven theorems on:

- Counting number fields with bounded discriminant.
- Matrices arising from finite field analogues of hypergeometric functions.
- Elliptic curves and congruent numbers.

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