Congruent numbers and local polynomials

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Vanderbilt University

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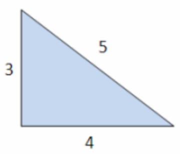

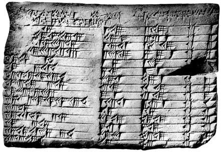

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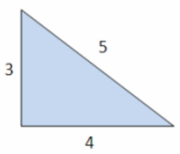

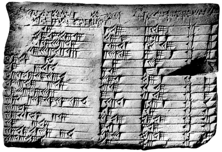

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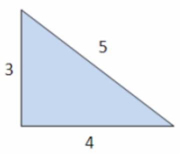

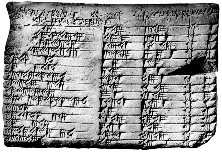

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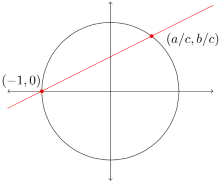

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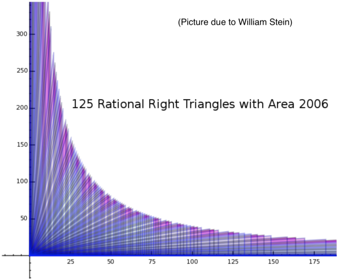

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- Zagier: the "simplest" triangle showing this has hypotenuse: $\frac{224403517704336969924557513090674863160948472041}{8912332268928859588025535178967163570016480830}$ (!!)

Connections to deeper theory

- There is a one-to-one correspondence:

$$\{(a,b,c) : \frac{ab}{2} = n, a^2+b^2 = c^2\} \leftrightarrow \{(x,y) : y^2 = x^3-n^2x, y \neq 0\}.$$

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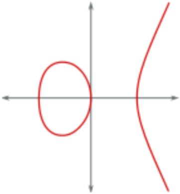

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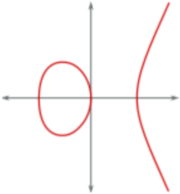

- Birch and Swinnerton-Dyer conjecture $\implies$ n is congruent if and only if the central L-value vanishes:

$$L(E,1) = 0.$$

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- Other formulas for $L(E_n,1)$ given by B-SD, for CM curves.
- We will give alternate formulas which include some non-CM cases and have analogies with classical formulas.

Classical results of Dirichlet and Gauss

- For $\chi_d := \left(\frac{d}{\cdot}\right)$, the **Dirichlet L-series** is

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- Gauss gave formulas like for $d \equiv 3 \pmod 8$:

$$h(-d) = \sum_{x^2+y^2+z^2=d} 1.$$

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Sample Theorem (Ehlen, Guerzhoy, Kane, R.)

Suppose that $D < 0$, $|D| \equiv 3 \pmod 8$, $3|D| \neq \square$. *Set*

$$\chi(a,b,c) := \begin{cases} \left(\frac{-3}{a}\right) & \text{if } 3 \nmid a, \\ \left(\frac{-3}{c}\right) & \text{if } 3 | a. \end{cases}$$

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Then, assuming BSD, $|D|$ *is congruent iff*

$$\sum_{\substack{b^2-4ac=-3D \\ c>0>a \\ 32|a}} \chi(a,b,c) = \sum_{\substack{b^2-4ac=-3D \\ a+3b+9c>0>a \\ 32|a}} \chi(a,b,c).$$

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- This was proven by Genz.

Behind the proofs

- Special functions introduced by Zagier:

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- We also need **cycle integrals** of modular forms (C_Q is a semicircle determined by Q):

$$r_{k,N}(f; D_0, |D|) := \sum_{[a,b,c]\in\Gamma_0(N)\backslash\mathcal{Q}_{DD_0},\ N|a} \chi_{D_0}(a,b,c) \times \int_{C_Q} f(\tau)(a\tau^2 + b\tau + c)^{k-1} d\tau.$$

Connection to L-values

Theorem (Kohnen)

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- Thus, if $L(f \otimes \chi_{D_0}, k) \neq 0$, then

$$L(f \otimes \chi_D, k) = 0 \iff \langle f, f_{k,N,D,D_0} \rangle = 0.$$

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- By Stokes' Theorem, to compute $\langle f, f_{k,N,D,D_0} \rangle$, take a "lift" under operator $\xi_{2-k} := 2i\,\mathrm{Im}(\tau)^{2-k}\overline{\frac{\partial}{\partial \overline{\tau}}}$.

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- The "natural" lift has discontinuities from local polynomials.

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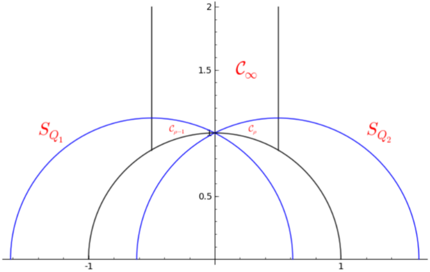

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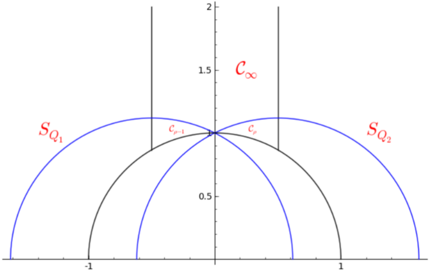

- The modular forms of weight -2 which are locally polynomial with "jumps" across the blue semicircles are ($\alpha, \beta \in \mathbb{C}$):

$$\begin{cases} \alpha & \text{if } \tau \in \mathcal{C}_\infty, \\ \beta\left(\tau^2 - \tau + 1\right) & \text{if } \tau \in \mathcal{C}_\rho, \\ \beta\left(\tau^2 + \tau + 1\right) & \text{if } \tau \in \mathcal{C}_{\rho-1}. \end{cases}$$

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