Benford's Law for Coefficients of Modular Forms

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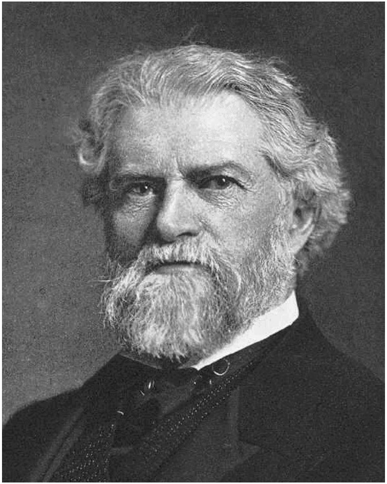
Simon Newcomb - 1881



Frank Benford

col.	title	1	2	3	4	5	6	7	8	9
A	Rivers, Area	31.0	16.4	10.7	11.3	7.2	8.6	5.5	4.2	5.1
B	Population	33.9	20.4	14.2	8.1	7.2	6.2	4.1	3.7	2.2
C	Constants	41.3	14.4	4.8	8.6	10.6	5.8	1.0	2.9	10.6
D	Newspapers	30.0	18.0	12.0	10.0	8.0	6.0	6.0	5.0	5.0
E	Specific Heat	24.0	18.4	16.2	14.6	10.6	4.1	3.2	4.8	4.1
F	Pressure	29.6	18.3	12.8	9.8	8.3	6.4	5.7	4.4	4.7
G	H.P. Lost	30.0	18.4	11.9	10.8	8.1	7.0	5.1	5.1	3.6
H	Mol. Wgt.	26.7	25.2	15.4	10.8	6.7	5.1	4.1	2.8	3.2
I	Drainage	27.1	23.9	13.8	12.6	8.2	5.0	5.0	2.5	1.9
J	Atomic Wgt.	47.2	18.7	5.5	4.4	6.6	4.4	3.3	4.4	5.5
K	$n^{-1}, \sqrt{n}$	25.7	20.3	9.7	6.8	6.6	6.8	7.2	8.0	8.9
L	Design	26.8	14.8	14.3	7.5	8.3	8.4	7.0	7.3	5.6
M	Reader's Digest	33.4	18.5	12.4	7.5	7.1	6.5	5.5	4.9	4.2
N	Cost Data	32.4	18.8	10.1	10.1	9.8	5.5	4.7	5.5	3.1
O	X-Ray Volts	27.9	17.5	14.4	9.0	8.1	7.4	5.1	5.8	4.8
P	Am. League	32.7	17.6	12.6	9.8	7.4	6.4	4.9	5.6	3.0
Q	Blackbody	31.0	17.3	14.1	8.7	6.6	7.0	5.2	4.7	5.4
R	Addresses	28.9	19.2	12.6	8.8	8.5	6.4	5.6	5.0	5.0
S	$n^1, n^2 \cdots n!$	25.3	16.0	12.0	10.0	8.5	8.8	6.8	7.1	5.5
T	Death Rate	27.0	18.6	15.7	9.4	6.7	6.5	7.2	4.8	4.1

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- Similar results hold for any base k.

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- $a(n)$ is Benford if and only if $\log_k(a(n))$ is uniformly distributed mod 1 for all k.

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x	$d = 1$	2	3	4	5	6	7	8	9
10^2	0.33	0.16	0.14	0.09	0.07	0.06	0.07	0.05	0.03
10^3	0.305	0.177	0.127	0.094	0.076	0.068	0.057	0.052	0.044
10^4	0.302	0.177	0.126	0.096	0.078	0.067	0.057	0.051	0.046
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
∞?	0.301	0.176	0.125	0.097	0.079	0.067	0.057	0.051	0.046

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- Why does this hold for $p(n)$?
- $p(n)$ is just one example of an infinite class of modular forms.

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- $\lim_{n\to\infty} \frac{D^{(h)} \log b(n)}{c^{(h)}(n)} = 0$

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- Corollary: $f(n)$ is uniformly distributed mod 1 whenever $f^{(k)}$ is monotonic and

$$\lim_{x \to \infty} f^{(k)}(x) = 0 \text{ and } \lim_{x \to \infty} x|f^{(k)}(x)| = \infty.$$

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Then $p(n)$ is good and hence Benford.

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Then recall that $f(z)$ is a weakly holomorphic modular form if its poles are supported at the cusps of Γ.

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Suppose that $M(z)$ is a weakly holomorphic modular form of weight $\frac{1}{2} \geq k \in \frac{1}{2}\mathbb{Z}$ with integral Fourier coefficients and at least one pole.
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$$\sum_{n=0}^{\infty} \frac{q^{n^2+an}}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+a+1})(1-q^{5n+4-a})},$$

for $a = 0, 1$.

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- Note that

$$I_\alpha(x) \sim \frac{e^x}{\sqrt{2\pi x}} \cdot (1 + \frac{(1-2\alpha)(1+2\alpha)}{8x} + \dots)$$

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Then consider for the case $p(n)$,

$$p(n) = 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right)$$

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The following table illustrates the first three digits of $p(n)$ in base 2

x	$d = 100$	d=101	d=110	d=111
200	0.285	0.270	0.205	0.225
400	0.308	0.273	0.209	0.205
600	0.313	0.267	0.217	0.198
800	0.314	0.263	0.219	0.201
1000	0.315	0.262	0.220	0.200
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