Conjectures of Andrews on partition-theoretic q-series

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- States of black holes

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- "Algebra" of adjectives: weakly, quasi, meromorphic, almost...

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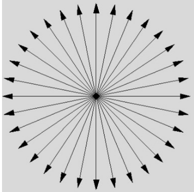

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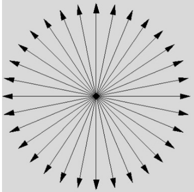

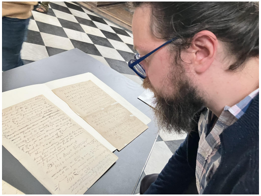

Finding modular forms in the sea of q-series

Easy Lemma (One-term Expansion)

If $f(q)$ is a (weakly hol.) modular form of weight k on a congruence subgroup, then there are numbers a, b such that

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- Mock modular forms are "near misses" off by just one term.

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- Nahm considered functions like

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Theorem (Zagier)

In the family

$$f_{A,B,C}(q) := \sum_{n\geq 0} \frac{q^{An^2+Bn+C}}{(q)_n},$$

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- Zagier proved that all others fail the One-term Expansion; none satisfy a two-term expansion of mock modular forms.

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- Famous Problem of Andrews: Prove modularity of the representation $P(q) = \sum_{n\geq 1} \frac{q^{n^2}}{(q)_n^2}$ **directly** without first proving a $\sum = \prod$ identity. Very hard, but you would immediately "suspect" modularity from radial asymptotics!

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where $\eta(\tau)$ is the Dedekind eta function, a MF of weight $1/2$.

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Theorem (Hardy–Ramanujan)

As $n \to \infty$ we have

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}}$$

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Key Idea (Cauchy's Theorem)

The coefficients $c(n)$ of a Fourier expansion $C(q) = \sum_{n\geq 0} c(n)q^n$ can be recovered as

$$c(n) = \frac{1}{2\pi i}\int_C C(q)q^{-n}\frac{dq}{q}$$

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- In many applications, the pole at $q = 1$ gives the largest growth and we call it the dominant pole.

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- This is much more flexible in dealing with non-modular functions.

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$$\sigma(q) := \sum_{n\geq 0} \frac{q^{\frac{n(n+1)}{2}}}{(-q;q)_n} =: \sum_{n\geq 0} S(n)q^n$$

$$= 1+q-q^2+2q^3-2q^4+q^5+q^7-2q^8+2q^{10}-q^{12}-2q^{13}+O(q^{14}).$$

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- No $|S(n)|$ for $n \leq 1600$ is ≥ 4, but terms can exceed 10^{13}.

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- Generating function version: indefinite theta function

$$q\sigma(q^{24}) = \sum_{a>6|b|} \left(\frac{12}{a}\right)(-1)^b q^{a^2-24b^2}.$$

Even deeper structure

- Cohen: σ has a friend,

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- Zwegers: These are analogues of the mock theta functions in his thesis; give "mock Maass theta functions."
- Zagier: These are *period functions* of the Maass waveform, and give *quantum modular forms.*

Other functions

- Another function from the Lost Notebook:

$$v_1(q) := \sum_{n \geq 0} \frac{q^{\frac{n(n+1)}{2}}}{(-q^2;q^2)_n} =: \sum_{n \geq 0} V_1(n) q^n.$$

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- The $V_1(n)$ count the difference between the number of odd-even partitions of n with rank $\equiv 0 \pmod 4$ and $\pmod 2$ $\pmod 4$.

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For almost all n, $V_1(n)$, $V_1(n+1)$, $V_1(n+2)$ *and* $V_1(n+3)$ *are two positive and two negative numbers.*

Data

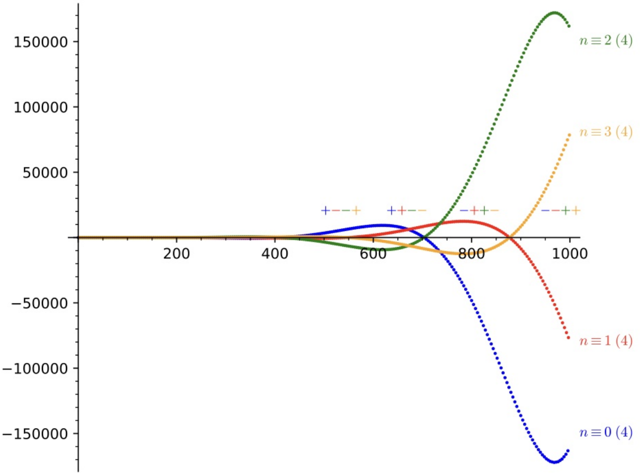

FIGURE 1. $V_1(n)$ for $n = 1, \ldots, 1000$

Main Result

Theorem (Folsom, Males, R., Storzer (2023))

The two conjectures of Andrews above are true.

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1. Determine radial asymptotics for the generating function.
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3. Interpret the Fourier coefficients.

Radial limits: the "easy case"

Lemma

Let $\zeta_N := e^{2\pi i/N}$*. For any root of unity* ζ_m^ℓ *with* $\gcd(\ell, m) = 1$ *and* $4 \nmid m$*, we have that*

$$v_1(\zeta_m^\ell) = 2\sum_{s=0}^{m-1} \frac{\zeta_{2m}^{\ell s(s+1)}}{(-\zeta_m^{2\ell}; \zeta_m^{2\ell})_s} = O(1).$$

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This is just some number, so we only need to worry about 4-mth roots of unity.

Radial limits: the "hard case"

Theorem (Folsom, Males, R., Storzer (2023))

If $4|m$, *then as* $z \to 0$ *(in a fixed cone in the right half-plane),*

$$v_1(\zeta e^{-z}) = e^{\frac{16V}{zm^2}} \sqrt{\frac{2\pi i}{z}} \left(\gamma^+_{(\alpha)} + O(|z|)\right) + e^{\frac{-16V}{zm^2}} \sqrt{\frac{2\pi i}{-z}} \left(\gamma^-_{(\alpha)} + O(|z|)\right).$$

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Here using Bloch-Wigner dilogarithm: $V = \mathrm{D}(e(1/6))\frac{i}{8}$, *the gamma numbers are, e.g,:*$\gamma^+ := \gamma^+_{(1/4)} = \gamma^-_{(3/4)} = \frac{1}{2\sqrt[4]{3(2-\sqrt{3})}}$ *and* $\gamma^- := \gamma^-_{(1/4)} = \gamma^+_{(3/4)} = \frac{1}{2\sqrt[4]{3(2+\sqrt{3})}}$.

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Theorem (Folsom, Males, R., Storzer (2023))

If $4|m$, then as $z \to 0$ (in a fixed cone in the right half-plane),

$$v_1(\zeta e^{-z}) = e^{\frac{16V}{zm^2}} \sqrt{\frac{2\pi i}{z}} \left(\gamma^+_{(\alpha)} + O(|z|)\right) + e^{\frac{-16V}{zm^2}} \sqrt{\frac{2\pi i}{-z}} \left(\gamma^-_{(\alpha)} + O(|z|)\right).$$

Here using Bloch-Wigner dilogarithm: $V = \mathrm{D}(e(1/6))\frac{i}{8}$, the gamma numbers are, e.g,:$\gamma^+ := \gamma^+_{(1/4)} = \gamma^-_{(3/4)} = \frac{1}{2\sqrt[4]{3(2-\sqrt{3})}}$ and $\gamma^- := \gamma^-_{(1/4)} = \gamma^+_{(3/4)} = \frac{1}{2\sqrt[4]{3(2+\sqrt{3})}}$.

In particular, the largest growth is at $\pm i$ (when $m = 4$).

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- Taking care of various branch cuts and poles/residues, make some changes of variable to massage the integral into a nicer form.
- Split into integral pieces, each of which should have different properties.
- Use the saddle-point method to determine the asymptotic behavior of the function toward fourth roots of unity.

"Taste" of the proof for $m = 4$

- Split up: $v_1^{[j]}(q) = \sum_{0 \leq n \equiv j \pmod 2} q^{n(n+1)/2}/(-q^2;q^2)_n$.

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$$\frac{-1}{2i}\frac{1}{(-q^2;q^2)_\infty}\int_{L_\infty} e^{\pi i s/4} e^{-zs(s+1)/2}(-e^{-2sz}q^2;q^2)_\infty \frac{1}{2\sin(\pi s/2)}$$

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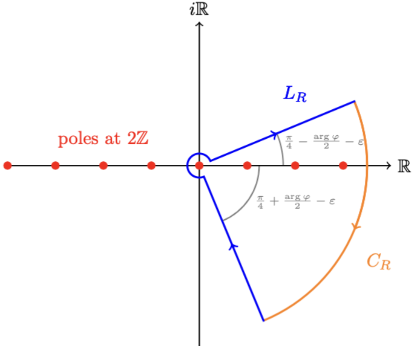
Here (φ specifies an angle we approach the root of unity at)



"Essence" of the saddle point method

- Suppose we have an integral of the shape:

$$\int_{\Gamma} f(z)e^{A \cdot g(z)}dz,$$

where f, g are holomorphic functions and $\Gamma \subseteq \mathbb{C}$ is a contour.

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- We want to estimate integrals of this shape as $A \to \infty$.
- Since f, g are holomorphic, we can deform the contour without changing the value of the integral (or if we have poles, we pick up residues).
- The points where the real part of $g(z)$ is maximized and the imaginary part of $g(z)$ is constant are called **saddle points**, and are zeros of $g'(z)$.

Saddle point method (cont.)

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- Then the integral can be written in terms of Gaussian-like integrals.
- These integrals may then be approximated by well-known means for large values of A.

Asymptotics for Fourier coefficients

Theorem (Folsom, Males, R., Storzer (2023))

As $n \to \infty$ we have

$$V_1(n) = (-1)^{\lfloor \frac{n}{2} \rfloor} \frac{e^{\sqrt{2|V|n}}}{\sqrt{n}} (\gamma^+ + (-1)^n \gamma^-) \times \left(\cos(\sqrt{2|V|n}) - (-1)^n \sin(\sqrt{2|V|n}) \right) \left(1 + O\left(n^{-\frac{1}{2}} \right) \right) + O\left(n^{-\frac{1}{2}} e^{\sqrt{\frac{|V|n}{2}}} \right).$$

Ideas of the proof

Write

$$V_1(n) = \frac{1}{2\pi i} \int_C \frac{v_1(q)}{q^n} \frac{dq}{q}.$$

Now let

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C-C_1-C_2},$$

where C_1 is a major arc around i, C_2 is a major arc around $-i$, and everything else is a minor arc.

Major arcs

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- Make the change of variable $q = ie^{-z}$ and parameterize where z runs from $\lambda + i\delta$ to $\lambda - i\delta$, to obtain

$$M_1(n) = -\frac{(-i)^n}{2\pi i} \int_{\lambda+i\delta}^{\lambda-i\delta} \frac{v_1\left(ie^{-z}\right)}{e^{-zn}} dz$$

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- This can be analyzed by the saddle-point method again.

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Principle

If you write a paper using the Circle Method, it always takes longer than you expect to carry out the details. Even if you expect it to take longer than you expect!

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- Thus we may bound the entire error term $E(n)$ by the contribution from the 8-th order roots of unity multiplied by the length of the integral, which is less than 2π.
- We then perform another saddle point analysis.

Sign pattern explanation (Conjecture 2)

- Asymptotics for $V_1(n)$ reduce us to study signs of

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$$\lim_{x\to\infty} \cos(a\sqrt{x+1}) - \cos(a\sqrt{x}) = \lim_{x\to\infty} \sin(a\sqrt{x+1}) - \sin(a\sqrt{x}) = 0.$$

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Conjecture (Andrews)

For $n \geq 5$ *there is an infinite sequence* $N_5 = 293, N_6 = 410, N_7 = 545, N_8 = 702, \ldots, N_n \geq 10n^2, \ldots$ *such that* $V_1(N_n), V_1(N_n + 1), V_1(N_n + 2)$ *all have the same sign.*

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The numbers $|V_1(N_n)|$, $|V_1(N_n + 1)|$, $|V_1(N_n + 2)|$ contain a local minimum of the sequence $|V_1(j)|$.

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- Milnor $\implies$ $|V| = \frac{9\sqrt{3}\zeta_{\mathbb{Q}(\sqrt{-3})}(2)}{16\pi^2}$.

More on these sorts of constants

- Siegel-Klingen: Used Hilbert modular forms to show that $\zeta_K(2n) \in \sqrt{|\operatorname{disc}(K)|}\pi^{2kN}\mathbb{Q}$ for $n \in \mathbb{N}$, K totally real.

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- Is this a hint of a modular object involving $\mathbb{Q}(\sqrt{-3})$???

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- We prove, or at least "explain" modulo hard irrationality questions, the conjectures of Andrews on V_1. There are additional functions with similar conjectures in his paper!

Thank you!!