WORKSHEET 3: CRITIQUING PROOFS

MATH 2106-D

We have discussed what it means to be rigorous and precise when you prove a mathematical statement. However, in order for a proof to be a proper proof, more than a rigorous understanding of why something is true is required. In particular, doing and sharing proofs is a *social* activity, and something counts as a good proof only when someone else can read and understand each of your arguments clearly. Since the goal of this class is to learn to write good proofs (and you need to do so to do well on the homeworks and exams), you need to learn to explain yourself clearly.

Just like to learn what is a rigorous proof you must learn what doesn't count as rigorous reasoning, to learn what is a well-written proof, you must learn what bad writing is. Below is a list of tips which you should keep in mind when writing, which includes some of the most common problems for beginning proof-writers.

(1) Always write complete, grammatically correct sentences with punctuation. For example, even when you end a sentence with a formula, that formula should still end in a period, and sentences should always start with a capital letter. Just as in other subjects, grammar and the use of full sentences are needed to make your writing understandable.

(2) Don't start sentences with a mathematical symbol. For example, instead of saying "$\forall x \in \mathbb{R}$, we therefore have that..," you could say "Thus, for any $x \in \mathbb{R}$, we have that." In particular, use connecting words like "thus" or "hence" instead of simply writing $\implies$ between every line or just writing a string of sentences without connections. Segues and small connecting words make a huge difference in readability and comprehension of a proof. While using math symbols in this way is great for shorthand in working through a problem at home or at the board (when you are in the room to explain each of the steps in person to your audience), when writing a proof on paper which anyone with the appropriate background should be able to follow in private, this is not good style.

(3) Don't use math symbols as verbs or to fill in for English parts of speech. For example, don't say "Thus, this is = to..." or "Thus, this number $\in$ the set X."

(4) Include words between successive math symbols, as cutting them out can often lead to confusing or hard to read, expressions. For example, instead of writing "Since $a|1, a = \pm 1$," you could say "Since $a|1$, we have that $a = \pm 1$."

(5) Don't introduce notation or symbols that aren't used. For example, if you write "No natural number n is irrational." but then never refer to the variable name n again, then you should have written the simpler sentence "No natural number is irrational."

(6) When introducing notation, define it as soon as it is used. For example, if you start using a variable name x but didn't say kind of thing x is, or why you have introduced it, you can leave a reader behind or to guess what you are doing. You can use notation which is standard without redefining it. For example, you don't have to explain what the symbol $\subseteq$ means, as you can expect your reader to know common mathematical language. If you write "Since m is a multiple of

3, we have $m = 3n$," a reader will likely "know what you mean," but this is still not a replacement for explaining what the number n is. For example, you could correct this writing by saying "Since m is a multiple of 3, there is an integer n for which $m = 3n$."

(7) Use the "royal we," as if you are guiding the reader through the proof and reading it together. That is, you shouldn't use "I" or "my" at all, and should instead use "we" or "one" (as in "one can show that...").

For the following two "proofs," write down comments about problems with the proofs, or with their writing style, and discuss with your neighbors. How many errors and examples of poor writing can you find? You can write on this sheet and write down a number 1-6 near any place where one of the numbered "faux pas" above takes place, and write comments about any other problems with the proofs you find.

(1)

Theorem. *For any natural number n, let $p_1 = 2,\ p_2 = 3,\ p_3 = 5, \dots, p_n$ be the first n prime numbers. Then $p_1 \cdot p_2 \cdot \dots \cdot p_n + 1$ is a prime number.*

Proof. $p_1 \cdot p_2 \cdot \dots \cdot p_n + 1$ is what we want to study. like in the proof that there are infinitely many primes

$$p_j | N,$$

some j, so since $N = p_1 \cdot p_2 \cdot \dots \cdot p_n + 1$
$\implies N \equiv 0 \pmod{p_j} \implies 1 = N - p_1 \cdot p_2 \cdot \dots \cdot p_n = N - p_j \cdot p_1 \cdot p_2 \cdot p_{j-1} \cdot p_{j+1} \dots \cdot p_n \equiv 0 \pmod{p_j}$. Its impossible for 1 to be $\equiv 0$ modulo an integer n that's ≥ 2 as this $\implies$ an impossible divisibility condition. Thus, we have reached a contradiction, and so p_j never $| N$. Hence, its a prime p. □

[N.B. This "theorem" is false, as one can find a counterexample: $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30,031 = 59 \cdot 509$ is composite. Thus, there must be something wrong with the proof.]

(2)

Theorem (Arithmetic-Geometric Mean Inequality)**.** *For any non-negative real numbers x and y, we have*

$$\sqrt{xy} \leq \frac{x+y}{2}.$$

Proof. $(x-y)^2$ is non-negative, since that's true $\forall \alpha$. So I can write that number as ≥ 0, and when you add $4xy$ to it, you get

$$4xy \leq x^2 + 2xy + y^2$$

here notice that I expanded out the polynomial $P(x, y) = (x-y)^2 + 4xy = x^2 - 2xy + y^2 = x^2 + 2xy + y^2$. So, now if $\alpha \leq \beta$, $\sqrt{\alpha} \leq \sqrt{\beta}$, as it is an increasing function, since the derivative is always > 0 where it makes sense. The lines above $\Longrightarrow$

$$2\sqrt{xy} \leq (x+y)$$

because you recall that the square of the right hand side is the right hand side in the inequality above. This is now pretty much the same thing as is claimed. $\square$

(3) Rewrite the above proof of the Arithmetic-Geometric Mean Inequality, and show it to your neighbors. Then make comments on your neighbors' proofs, and try to come up with the best, most clearly written proof you can together.