WORKSHEET 1: PRACTICE ON SET THEORY

MATH 2106-D

(1) Let $X = \{1, 5, 8\}$ and $Y = \{1, a, b\}$. Write down the following sets:
  (a) $X \cap Y$.
  (b) $X \cup Y$.
  (c) $\mathcal{P}(X \cap Y)$.
  (d) $X \times Y$

(2) If $\mathcal{I}$ is a set, called an *index set*, and for each $\alpha \in \mathcal{I}$, we have a set A_α, then we can consider the union and intersection

$$\cup_{\alpha \in \mathcal{I}} A_\alpha = \{x | x \in A_\alpha \text{ for at least one } A_\alpha \text{ with } \alpha \in \mathcal{I}\},$$

$$\cap_{\alpha \in \mathcal{I}} A_\alpha = \{x | x \in A_\alpha \text{ for every } A_\alpha \text{ with } \alpha \in \mathcal{I}\}.$$

Determine what the following sets are, and explain your answer.
  (a) $\cap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)$
  (b) $\cup_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)$
  (c) $\cup_{a \in \mathbb{R}} \{(x, y, z) | x^2 + y^2 = a^2, z = a\}$

(3) Explain why $|A \cup B| = |A| + |B| - |A \cap B|$. Can you find a similar formula for $|A \cup B \cup C|$ in terms of cardinalities of A, B, C and intersections between these sets?

(***) Let X be the set of all sets that are not elements of themselves, that is,

$$X = \{A | A \text{ is a set, } A \notin A\}.$$

Explain why the existence of this "set" doesn't make logical sense. (Hint: Is $X \in X$?) This conundrum, known as **Russell's Paradox**, shows that in order to be 100% rigorous with set theory, one must be very careful when describing which sets are "allowed"; the standard resolution to this problem is to base set theory on a precise set of **axioms**, such as **ZFC**.

Handy notation cheat sheet:

- $a \in A$, read "a is in A," means that a is an element of A

- $A \subseteq B$, read "A is a subset of B," means that every element of A is also an element of B
- $A = B$, read "A equals B," if A and B have the same elements

- $|A|$ is the cardinality, or size of A, namely the number of its elements (if A is finite; if A is infinite, one often writes $|A| = \infty$ as a shorthand, although this isn't very precise)

- $\emptyset$ denotes the **empty set**, which is the only set with no elements

- $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of **natural numbers**

- $\mathbb{Z} = \{\dots, -3, -2, -3, 0, 1, 2, 3, \dots\}$ is the set of **integers** (whole numbers)

- $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, \text{ and } b \neq 0\}$ is the set of **rational numbers** (fractions)

- $\mathbb{R}$ is the set of **real numbers**

- $A \times B$ denotes the **Cartesian product** $A \times B = \{(a, b) \mid a \in A \ , b \in B\}$ consisting of **ordered pairs** of elements from A and B. The n-fold Cartesian product $A \times \dots \times A$ is denoted by A^n.

- The notations $\mathcal{P}(A)$ and 2^A both denote the **power set of** A, namely the set of all subsets of A. That is, $\mathcal{P}(A) = \{X \mid X \subseteq A\}$.

- The **union** of A and B is $A \cup B = \{x | x \in A \text{ or } x \in B\}$ (where *or* in math always means that one or the other or both conditions hold)

- The **intersection** of A and B is $A \cap B = \{x | x \in A \text{ and } x \in B\}$

- The **set difference** of A and B is $A \setminus B = A - B = \{x | x \in A \text{ and } x \notin B\}$

- If U is the universal set, then the **complement of** A is $A^c = \overline{A} = U \setminus A$ (i.e., the set of everything not inside of A, in whatever universal set you have given by the context)