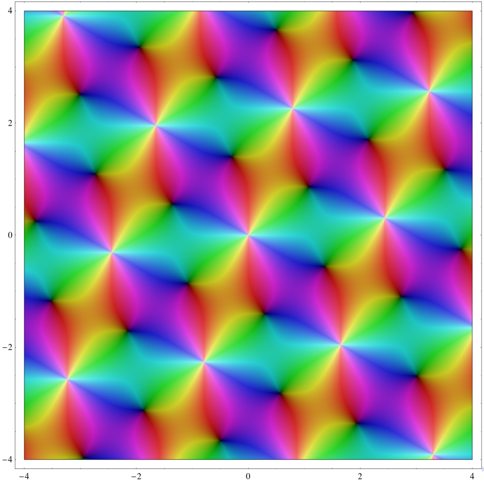
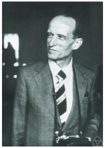
Modular Forms



Quotes

Martin Eichler: Five elementary arith. operations: $+$,-,$\times$, $\div$, MFs



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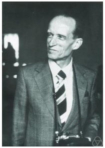
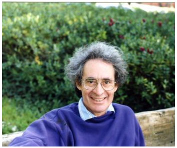


Barry Mazur: "Modular forms are functions... that are inordinately symmetric. They satisfy so many internal symmetries that their mere existence seem like accidents. But they do exist."



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- Modularity:
  1. $f|_k\gamma = f \quad \forall \gamma \in \Gamma \leq \mathrm{SL}_2(\mathbb{Z})$.
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- New Quanta article: `https://tinyurl.com/vv8mcjuw`
- "Algebra" of adjectives: weakly, quasi, meromorphic, almost...

First observations

- If $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then $\mathrm{SL}_2(\mathbb{Z}) = \langle S, T \rangle$.

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- Often in combinatorics (integer partitions, etc.), physics (statistical mechanics, string theory...), knot theory (volume conj.), want to determine *asymptotics* of sequences. Asymptotic method work if gen. fun. is modular in any way: $f|\gamma = f + g$, for g small, or $f|\gamma = g_1 + g_2$.

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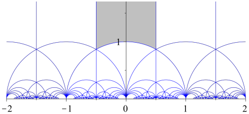
Lattice sums of functions that are essentially eigenfunctions of Fourier transf. give MFs. Example: $\theta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}$.

Application 1: Identities

- MFs are determined by values on a **fundamental domain**:



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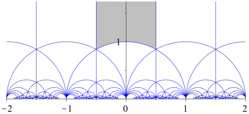

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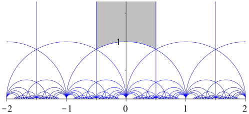

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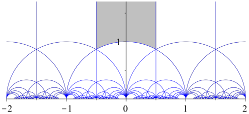

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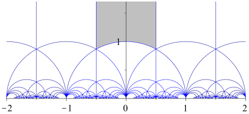

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- This means that checking identities is a **finite check.**
- Coeffs. of θ^4 count ways to write n as a sum of 4 squares, $r_4(n)$. There's a Poincaré series too, dimension $= 2 \rightsquigarrow$

$$r_4(n) = 8 \sum_{\substack{d|n \\ 4\nmid d}} d.$$

Similarly, for sums of 8 squares, $r_8(n) = 16 \sum_{d|n} (-1)^{n+d} d^3$.

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- *Infinitely many* of these convolution ident. from finite check.

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- Borcherds proof earned him a Fields Medal.

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- Integer partition function $p(n)$ counts number of ways to write n as a sum of natural numbers.

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which implies modularity.Hardy-Ramanujan used these transformations to prove

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}.$$

Application 4: Arithmetic Geometry

Theorem (Modularity Theorem of Wiles,Taylor-Wiles, et al)

For each elliptic curve $/\mathbb{Q}$, *there is a special modular form whose coeffs. determine the number of points of* E *over all finite fields.*

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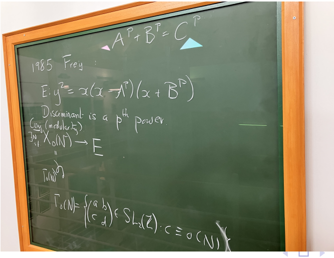

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- This theory "explains" my favorite number: $e^{\pi\sqrt{163}} = 262537412640768743.99999999999925\ldots \approx\in \mathbb{Z}$.

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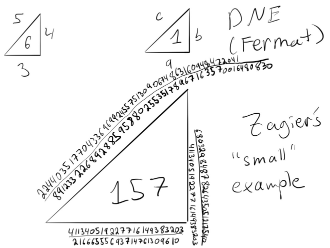

THANK YOU!!!