Cranks for colored partitions

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Partitions

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A *partition* of an integer n is a sequence of positive integers

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Example

The partitions of 4 are

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1,$$

so $p(4) = 5$.

Ramanujan congruences

Theorem (Hardy, Ramanujan, 1919)

For all $n \in \mathbb{N}$,

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

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Theorem (Atkin–Swinnerton-Dyer, 1954)

The ranks for $5n+4$ are equidistributed modulo 5, i.e.

$$N(0,5,5n+4) = N(1,5,5n+4) = \cdots = N(4,5,5n+4).$$

Similarly, the ranks for $7n+5$ are equidistributed modulo 7.

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Definition (Garvan, Andrews–Garvan, 1988)

For a partition λ, let $\ell(\lambda)$ denote the largest part of λ, $\omega(\lambda)$ denote the number of 1's in λ, and $\mu(\lambda)$ denote the number of parts of λ larger than $\omega(\lambda)$. The *crank* of λ is

$$\begin{cases} \ell(\lambda) & \text{if } \omega(\lambda) = 0 \\ \mu(\lambda) - \omega(\lambda) & \text{if } \omega(\lambda) > 0. \end{cases}$$

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Theorem (Andrews–Garvan, 1988)

The cranks for $5n+4$, $7n+5$, *and* $11n+6$ *are equidistributed modulo* 5, 7, *and* 11 *respectively.*

Reframing equidistribution

Let $M(m,n)$ denote the number of partitions of n with crank m. The generating function is ($\zeta = e^{2\pi i z}$, $q = e^{2\pi i \tau}$)

$$\mathcal{C}(z;\tau) := \sum_{n=0}^{\infty} \sum_{m=-n}^{n} M(m,n)\zeta^m q^n$$

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$$= q^{\frac{1}{24}}(\zeta^{-\frac{1}{2}} - \zeta^{\frac{1}{2}})\frac{\eta^2(\tau)}{\theta(z;\tau)}.$$

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For prime ℓ, let $\Phi_\ell(\zeta) = 1 + \zeta + \cdots + \zeta^{\ell-1}$.

Lemma

The equidistribution of the crank mod ℓ for $\ell n + \beta$ is equivalent to

$$\Phi_\ell(\zeta) \Big| [q^{\ell n+\beta}]\mathcal{C}(z;\tau).$$

Other partition congruences

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Theorem (Gritsenko–Skoruppa–Zagier)

Let R be an irreducible root system with a choice of positive roots R^+, and let $w = \frac{1}{2}\sum_{r\in R^+} r$. Then we have

$$\theta_R(z;\tau) := \eta(\tau)^{n-N} \prod_{r\in R^+} \theta\left(\frac{(r,z)}{h};\tau\right)$$

$$= \sum_{x\in w+W_{R,ev}} q^{\frac{(x,x)}{2h}} \sum_{g\in G_R} \mathrm{sn}(g) e\left(\frac{(gx,z)}{h}\right)$$

for all $\tau \in \mathbb{H}$ and $z \in \mathbb{C}\otimes W_R$. In particular, θ_R is a holomorphic Jacobi form in $J_{\frac{n}{2},\underline{R}}(\epsilon^{n+2N})$.

Important: $\sum = \prod$ formulas for various root systems.

Families of cranks

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For k odd, we try the form:

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1. Numerator: Only ϕ_R remains.
2. Denominator: Look what happens $\pmod{\Phi_\ell(\zeta)}$.

Which congruences can we explain?

Reminder: For k odd,

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If there is not a congruence for $k \equiv -14 \pmod{\ell}$, we define

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The first definition explains all congruences coming from $k \equiv -2, -4, -6, -8, -10 \pmod{\ell}$, while the second explains those coming from $k \equiv -2, -4, -6, -10, -14 \pmod{\ell}$.

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