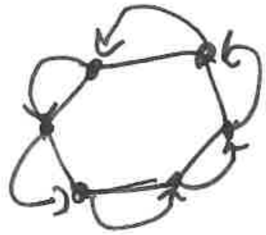
Sol^{ns} to extra Burnside's Lemma problems:

4). We list the elements of D_{12} and count fixed pts (the chemical attachments are a 3-coloring: put nothing, OH, or CH_3).

id: 3^6 (all possible colorings are invariant)

r: 3 (all must be the same attachment)

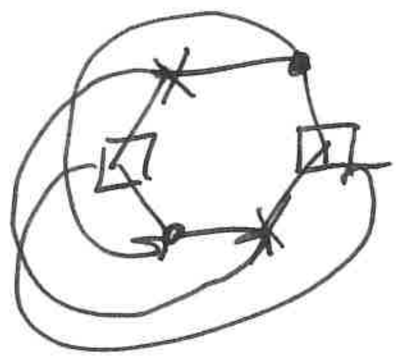


• denotes the cycle

r^2: 3^2 (there are 2 3-cycles)

•, x denote cycles

r^3: 3^3 (there are 3 2-cycles)



x, •, □ denote cycles.

r^4: 3^2 (similar to r^2)

r^5: 3^1 (similar to r^1):

Reflections

A). 3 across lines through opposite vertices

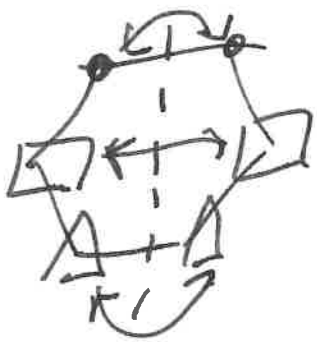
(cycles: •, Δ, $\square$, x).

in each, there are 2 1-cycles

2 2-cycles, $\rightsquigarrow$ 4 free choices for labelling

Contributing $3^4 \cdot 3$ total fixed pts.

B). 3 across lines thru opposite midpts:



Cycles: •, $\square$, Δ.

there are 3 2-cycles

$\rightsquigarrow$ 3^3 fixed pts of each reflection

$\rightsquigarrow$ contributes $3^3 \cdot 3$ total fixed pts

Using Burnside, $\rightsquigarrow$

$$\text{average \# fixed pts} = \frac{1}{12} \cdot \left(3^6 + 3 + 3^2 + 3^3 + 3^2 + 3^2 + 3^1 + 3 \cdot 3^4 + 3 \cdot 3^3\right) = \underline{92}$$

$\rightsquigarrow$ 92, "chemically different" compounds.

5) • The identity fixes all patterns of X's, O's.
which is $\binom{9}{4} = \underline{126}$. (pick 4 boxes to put the O's, fill rest @ X's).

• $90°$ rotation counter-clockwise:

This breaks the grid into 3 orbits:
center, corners, middle:

xx	x	xx
x	•	x
xx	x	xx

We must place x in middle (as # X's $\equiv 1 \pmod 4$), then either all "x" spots or "xx" spots above.
$\rightarrow \underline{2}$ fixed pts.

• $90°$ rotation clockwise: similarly, $\exists$ 2 fixed pts.

• $180°$ rotation: 1 orbit of size 1 (center)
4 orbits of size 2 labelled by # of X's:

xx	x	xxx
xxxx	•	xxxx
xxx	x	xx

# of invariant patterns:
must put X in center (# X's $\equiv 1 \pmod 2$), and at all pts in 2 of the 4 orbits

total $\# = \binom{4}{2} = \underline{6}$ fixed pts.

- reflections:

A). 2 through lines connecting opposite ~~vertices~~ midpts of square:

x	y	x
xx	yy	xx
xxx	yyy	xxx

$\rightsquigarrow$ orbits are 3 sets of 2 (x, xx, xxx),
3 sets of size 1: (y, yy, yyy)

<u>Possibilities</u>: Must have an odd # of x's on line of ~~#~~ symmetry)

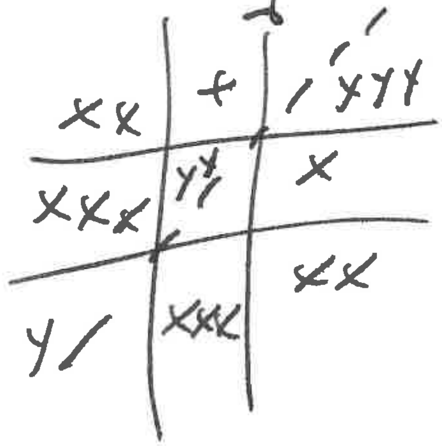
i): ~~x~~ Put 1 x on line of symm.
$\rightsquigarrow$ 3 options for which ~~x~~ y, yy, yyy to choose on symmetry line, then pick 2/3 orbits off the line

$\rightsquigarrow 3 \cdot \binom{3}{2} = $ 9 options.

ii): Put 3 x's on line of symmetry (all y, yy, yyy are x), pick $\underline{1}$ orbit off line of symmetry $\rightsquigarrow$ 3 options

Overall, 12 invariant patterns for each reflection.

B) Reflections through two ~~mid pts~~ opposite vertices of square (2 of them)



The argument works exactly the same as in (A):

the cycle lengths are all the same.

So each of these gives 12 fixed pts too.

⟶ # of different patterns, up to symmetry, by Burnside, 3

$$\frac{\underbrace{126}_{\text{identity}} + \underbrace{2+2}_{90^\circ,\,270^\circ} + \underbrace{\cancel{12}6}_{180^\circ} + \underbrace{12+12+12+12}_{4\text{ reflections}}}{\underbrace{8}_{\text{total \# of symmetries}}} = \underbrace{23}_{\text{total arrangements}}$$