Ramanujan’s Deathbed Letter

Larry Rolen

Emory University
The great anticipator of mathematics

Srinivasa Ramanujan (1887-1920)
Dear Hardy,

“I am extremely sorry for not writing you a single letter up to now. I discovered very interesting functions recently which I call “Mock” \(\vartheta\)-functions. Unlike the “False” \(\vartheta\)-functions (partially studied by Rogers), they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples.”

Ramanujan, January 12, 1920.
The first example

\[ f(q) = 1 + \frac{q}{(1 + q)^2} + \frac{q^4}{(1 + q)^2(1 + q^2)^2} + \cdots \]
Ramanujan's deathbed letter

Freeman Dyson (1987)

"The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. . . . This remains a challenge for the future..."
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“The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered....This remains a challenge for the future...”
In his Ph.D. thesis under Zagier ('02), Zwegers investigated:

Ramanujan's mock theta functions are holomorphic parts of weight 1/2 harmonic Maass forms.
The future is now

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**Theorem**

Ramanujan’s mock theta functions are holomorphic parts of weight 1/2 harmonic Maass forms.
**Notation.** Throughout, let $z = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$. 
Defining Maass forms

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**Hyperbolic Laplacian.**

\[
\Delta_k := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).
\]
Harmonic Maass forms

“Definition”

A harmonic Maass form is any smooth function $f$ on $\mathbb{H}$ satisfying:

1. For all $A \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset \text{SL}_2(\mathbb{Z})$ we have
   $$f \left( \frac{az+b}{cz+d} \right) = t^{k} f(z)$$

2. We have that $\Delta^k f = 0$.
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Remark:

Modular forms are holomorphic functions which satisfy (1).
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Remark

Modular forms are holomorphic functions which satisfy (1).
HMFs have two parts ($q := e^{2\pi iz}$)

**Fundamental Lemma**

If $f \in H_{2-k}$ and $\Gamma(a, x)$ is the incomplete $\Gamma$-function, then

$$f(z) = \sum_{n \gg -\infty} c_f^+(n)q^n + \sum_{n < 0} c_f^-(n)\Gamma(k - 1, 4\pi |n|y)q^n.$$ 

\uparrow

**Holomorphic part** $f^+$  \quad \uparrow

**Nonholomorphic part** $f^-$
Ramanujan’s deathbed letter

Maass forms

HMFs have two parts \((q := e^{2\pi i z})\)

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\[\uparrow\]

Holomorphic part \(f^+\)  Nonholomorphic part \(f^-\)

**Remark**

The mock theta functions are examples of \(f^+\).
So many recent applications

- $q$-series and partitions
- Modular $L$-functions (e.g. BSD numbers)
- Eichler-Shimura Theory
- Probability models
- Generalized Borcherds Products
- Moonshine for affine Lie superalgebras and $M_{24}$
- Donaldson invariants
- Black holes
- ...
Ramanujan's deathbed letter
Is there more?

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Ramanujan’s last letter.

- Asymptotics, near roots of unity, of “Eulerian” modular forms.
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- Raises one question and conjectures the answer.
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- Raises **one** question and conjectures the answer.
- Gives **one example** supporting his conjectured answer.
Ramanujan’s last letter.

- Asymptotics, near roots of unity, of “Eulerian” modular forms.

- Raises one question and conjectures the answer.

- Gives one example supporting his conjectured answer.

- Concludes with a list of his mock theta functions.
Ramanujan’s question

**Question (Ramanujan)**

*Must Eulerian series with “similar asymptotics” be the sum of a modular form and a function which is $O(1)$ at all roots of unity?*
The answer is *it is not necessarily so.*

When it is not so I call the function Mock D-function. I have not proved rigorously that it is not necessarily so. But I have constructed a number of examples in which it is not in conceivable to construct a D-function to cut out the singularities.
Ramanujan’s “Example”

If I have proved that if

$$f(q) = 1 + \frac{q^4}{(1+q)^2(1+q^2)^2} + \ldots$$

then

$$f(q) + (1-q)(1-q^3)(1-q^5)\ldots \in \mathbb{O}(1-2q+2q^2-2q^9+\ldots)$$

at all the

$$= \mathbb{O}(1)$$

at all the points

$$q\equiv -1, q^3\equiv -1, q^5\equiv -1, q^7\equiv -1, \ldots$$

and at the same time

$$f(q) \in (1-q)(1-q^3)(1-q^5)\ldots(1-2q+2q^2-\ldots)$$

$$= \mathbb{O}(1)$$

at all the points

$$q^2\equiv -1, q^4\equiv -1, q^6\equiv -1, \ldots$$

Also obviously

$$f(q) = \mathbb{O}(1)$$

at all the points

$$q\equiv 1, q^2\equiv 1, q^5\equiv 1, \ldots$$
Ramanujan’s “Near Miss Example”

Define the mock theta $f(q)$ and the modular form $b(q)$ by

\[
f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1 + q)^2(1 + q^2)^2 \cdots (1 + q^n)^2},
\]

\[
b(q) := (1 - q)(1 - q^3)(1 - q^5) \cdots \times (1 - 2q + 2q^4 - 2q^9 + \cdots).
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Conjecture (Ramanujan)

*If $q$ approaches an even order $2k$ root of unity, then*

$$f(q) - (-1)^k b(q) = O(1).$$
Ramanujan’s deathbed letter
Is there more?

Numerics
As \( q \to -1 \), we have

\[
\begin{align*}
  f(-0.994) &\sim -1 \cdot 10^{31}, \\
  f(-0.996) &\sim -1 \cdot 10^{46}, \\
  f(-0.998) &\sim -6 \cdot 10^{90},
\end{align*}
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\[ f(-0.998185) \sim -\text{Googol} \]
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Numerics continued...
Amasingly, Ramanujan’s guess gives:

<table>
<thead>
<tr>
<th>$q$</th>
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<tbody>
<tr>
<td>$f(q) + b(q)$</td>
<td>3.961...</td>
<td>3.969...</td>
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This suggests that

$$\lim_{{q \to -1}} (f(q) + b(q)) = 4.$$
As \( q \to i \)
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<th>$q$</th>
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This suggests that

$$\lim_{q\to i}(f(q) - b(q)) = 4i.$$
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Natural Questions
If he is right, then what are the mysterious $O(1)$ numbers in

$$
\lim_{q \to \zeta} (f(q) - (-1)^kb(q)) = O(1)?
$$
Finite sums of roots of unity.
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**Theorem (Folsom, Ono, Rhoades)**

If \( \zeta \) is an even \( 2k \) order root of unity, then

\[
\lim_{q \to \zeta} (f(q) - (-1)^k b(q)) = -4 \sum_{n=0}^{k-1} (1 + \zeta)^2 (1 + \zeta^2)^2 \cdots (1 + \zeta^n)^2 \zeta^{n+1}.
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**Theorem (Folsom, Ono, Rhoades)**

If $\zeta$ is an even $2k$ order root of unity, then

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**Remark**

This Theorem follows from “quantum” modularity.
“it is inconceivable to construct a \( \vartheta \) function to cut out the singularities of a mock theta function...”

Srinivasa Ramanujan
Ramanujan’s last words

“it is inconceivable to construct a $\vartheta$ function to cut out the singularities of a mock theta function…”

Srinivasa Ramanujan

“...it has not been proved that any of Ramanujan’s mock theta functions really are mock theta functions according to his definition.”

Bruce Berndt (2012)
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Theorem (G-Ono-Rolen (2013))

Ramanujan’s examples satisfy his own definition.
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Bruce Berndt (2012)

Theorem (G-Ono-Rolen (2013))

Ramanujan’s examples satisfy his own definition. More precisely, a mock theta function and a modular form never cut out exactly the same singularities.