Ranks, cranks, and new directions in partitions

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Recalling Definitions

Definition

An **integer partition** *of n is a sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ such that*

$$\lambda_1 + \ldots + \lambda_k = n.$$

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Definition (Andrews-Garvan, 1988)

$$crank(\lambda) := \begin{cases} \text{largest part of } \lambda & \text{if no 1's in } \lambda, \\ (\#\text{ parts larger than } \#\text{ of 1's}) - (\#\text{ of 1's}) & \text{else.} \end{cases}$$

Recall: equidistribution of ranks and cranks

Theorem (Conjecture of Dyson 1944, proven by Atkin and Swinnerton-Dyer in 1954)

We have

$$N(0,5;5n+4) = N(1,5;5n+4) = \ldots = N(4,5;5n+4).$$

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Theorem (Andrews-Garvan, 1988)

Cranks "explain" Ramanujan's congruences mod 5, 7, *and* 11.

Building to Stanton's Conjecutre

Elementary Fact

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Lemma

Let $f(\zeta)$ be a rational Laurent polynomial and ℓ be prime. Set $\widehat{f}_{r,\ell} := \sum_{j\equiv r \pmod{\ell}} [\zeta^j] f(\zeta)$. Then

$$\Phi_\ell | f(\zeta) \iff \widehat{f}_{r,\ell} = \widehat{f}_{\ell-1,\ell}, \quad r \in \{0,\ldots,\ell-2\}.$$

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Proof.

1. Multiply by a big power of ζ and use $\gcd(\zeta, \Phi_\ell(\zeta)) = 1$ to assume $f(\zeta) \in \mathbb{Q}[\zeta]$.

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3. If $f(\zeta) =: \sum_{j=0}^{n} a_j \zeta^j$,

$$f(\zeta_\ell) = \sum_{j=0}^{n} a_j \zeta_j^\ell = \sum_{r=0}^{\ell-1} \sum_{\substack{0 \le j \le n \\ j \equiv r \pmod{\ell}}} a_j \zeta_\ell^r = \sum_{r=0}^{\ell-1} \widehat{f}_{r,\ell} \zeta_\ell^r$$

$$= \sum_{r=0}^{\ell-2} \left(\widehat{f}_{r,\ell} - \widehat{f}_{\ell-1,\ell} \right) \zeta_\ell^r$$

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4. Claim follows as $\{1, \zeta_\ell, \ldots, \zeta_\ell^{\ell-2}\}$ is a basis for $\mathbb{Q}[\zeta]/\mathbb{Q}$.

Recalling Stanton's Conjecture

Definition (Stanton)

The **modified rank and crank** *are:*

$$rank^*_{\ell,n}(\zeta) := rank_{\ell n+\beta} + \zeta^{\ell n+\beta-2} - \zeta^{\ell n+\beta-1} + \zeta^{2-\ell n-\beta} - \zeta^{1-\ell n-\beta},$$

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where $\beta := \ell - \frac{\ell^2-1}{24}$.

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Conjecture (Stanton)

All of the following are Laurent polynomials with positive coefficients:

$$\frac{rank^*_{5,n}(\zeta)}{\Phi_5(\zeta)}, \quad \frac{rank^*_{7,n}(\zeta)}{\Phi_7(\zeta)}, \quad \frac{crank^*_{5,n}(\zeta)}{\Phi_5(\zeta)}, \quad \frac{crank^*_{7,n}(\zeta)}{\Phi_7(\zeta)}, \quad \frac{crank^*_{11,n}(\zeta)}{\Phi_{11}(\zeta)}.$$

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$$\left(\frac{1}{1-\zeta^\ell}\right)\mathrm{crank}^*_{\ell n+\beta}(\zeta)(1-\zeta).$$

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3. Thus, its enough to know that the coefficients of $\mathrm{crank^*}_{\ell n+\beta}(\zeta)$ are symmetric under $\zeta \mapsto \zeta^{-1}$. and unimodal.
4. Symmetry is direct form gen. fun. Reduced to finite check by Ji-Zang: $M(m-1,n) \geq M(m,n)$ if $n \geq 44$, $1 \leq m \leq n-1$.

□

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  4. When $k \equiv -4, -6, -8, -10, -14, -26 \pmod{\ell}$:

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An example

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- *Thus,*

$$\sum_{n \geq 0} p_{\ell t-3}(n) q^{8n+1} = q \frac{(q^8; q^8)_\infty^3}{(q^8; q^8)_\infty^{\ell t}}$$

$$\equiv \frac{\sum_{n \geq 0} (-1)^n (2n+1) q^{(2n+1)^2}}{(q^{8\ell}; q^{8\ell})_\infty^t} \pmod{\ell}.$$

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Theorem (Tripp-R.-Wagner 2020)

There is an infinite family of crank functions $C_k(z;\tau)$which explain "most" congruences of colored partitions.

Theta blocks

Definition

An eta quotient is a modular form of the form

$$\frac{\eta^{a_1}(b_1\tau)\cdot\ldots\cdot\eta^{a_k}(b_k\tau)}{\eta^{c_1}(d_1\tau)\cdot\ldots\cdot\eta^{c_k}(d_\ell\tau)}.$$

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Definition (Gritsenko-Skoruppa-Zagier)

Let $\vartheta_a(z) := \vartheta(az;\tau)$. Then a theta block is a holomorphic Jacobi form of the shape:

$$\frac{\vartheta_{a_1}(z)\cdot\ldots\cdot\vartheta_{a_k}(z)}{\vartheta_{b_1}(z)\cdot\ldots\cdot\vartheta_{b_\ell}(z)}\cdot\eta^n,\quad a_i, b_i \in \mathbb{N}\ , n\in\mathbb{Z}.$$

Examples of theta blocks

Example

We have the following Quintuple Product Identity:

$$\frac{\vartheta_2(z)}{\vartheta(z)}\eta = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{(6n+1)^2}{24}} \left(\zeta^{3n+\frac{1}{2}} + \zeta^{-3n-\frac{1}{2}}\right).$$

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Gritsenko-Skoruppa-Zagier defined the family of **theta quarks** *as*

$$\vartheta^*(z) := \frac{\vartheta_a(z)\vartheta_b(z)\vartheta_{a+b}(z)}{\eta} = -\sum_{m,n\in\mathbb{Z}} q^{\frac{m^2+mn+n^2}{3}}\zeta^{(a-b)m+an}.$$

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$$\phi\left(\frac{z}{c\tau + d}; \gamma\tau\right) = e\left(\frac{c\beta(z)}{c\tau + d}\right)(c\tau + d)^{k-\frac{h}{2}}\varepsilon^h(\gamma)\phi(z;\tau)$$

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$$\phi(z + x\tau + y; \tau) = e(\beta(x+y) - \tau\beta(x) - \beta(x,z))\phi(z;\tau).$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $x, y \in L$ and has Fourier expansion:

$$\phi(z;\tau) = \sum_{n \in \frac{h}{24}+\mathbb{Z}} \sum_{\substack{r \in L^\bullet \\ n \geq \beta(r)}} c(n,r) e(\beta(r,z)) q^n.$$

Eutactic stars

Definition

A **eutactic star of rank** N on a lattice $\underline{L}$ is a family s of non-zero vectors $s_j \in L^{\#}(1 \le j \le N)$ such that

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Definition

Let $G \subseteq O(\underline{L})$ with the property that for each $g \in G$ there exists a permutation σ of the indices $1 \leq j \leq N$ and signs $\varepsilon_j \in \{\pm 1\}$ such that $gs_j = \varepsilon_j s_{\sigma(j)}$ $\forall j$. Define the linear character $\mathrm{sn} : G \to \{\pm 1\}$ by

$$\mathrm{sn}(g) := \prod_j \varepsilon_j.$$

Eutactic stars

- The **shadow** of L is

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Definition

A eutactic star is G-**extremal on** $\underline{L}$ if there is exactly one G-orbit in $L^\bullet / L_{ev}$ whose elements have their stabilizers in the kernel of sn.

A product to sum theorem

Theorem (Gritsenko-Skoruppa-Zagier)

Let $\underline{L} = (L, \beta)$ be an integral lattice of rank n, let s be a G-extremal eutactic star of rank N on $\underline{L}$. Then there is a constant γ and a vector $w \in L^{\bullet}$ such that

$$\eta(\tau)^{n-N} \prod_{j=1}^{N} \theta(\beta(s_j, z); \tau) = \gamma \sum_{x \in w + L_{ev}} q^{\beta(x)} \sum_{g \in G} \mathrm{sn}(g) e(\beta(gx, z)).$$

In particular, the product on the left defines an element of $J_{\frac{n}{2}, \underline{L}}(\varepsilon^{n+2N})$.

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1. The roots span E_R.

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A **root system** R with associated Euclidean space E_R with inner product $(\cdot, \cdot)$ is a finite set of non-zero vectors (roots) that satisfy the following:

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3. For any two $r, v \in R$, we have $2\frac{(r,v)}{(r,r)} \in \mathbb{Z}$.
4. For any two $r, v \in R$, we have $s_r(v) := v - 2\frac{(r,v)}{(r,r)} \in R$.

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- Define the lattice

$$W_R := \left\{ x \in E_R : \frac{(x,r)}{h} \in \mathbb{Z} \text{ for all } r \in R \right\}$$

and set $\underline{R} := \left(W_r, \frac{(\cdot,\cdot)}{h} \right)$.

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- Let G_R be the **Weyl group**; the group generated by all of the s_r for $r \in R$.

Product to sum theorem for root systems

Theorem (Gritsenko-Skoruppa-Zagier)

Assume the previous notation. Then R^+ is a eutactic star on $\underline{R}$ and is extremal with respect to G_R.

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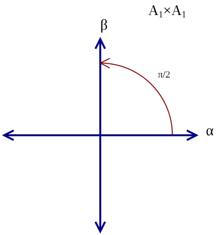
Theorem

Let R be an irreducible root system with a choice of positive roots R^+, and let $w = \frac{1}{2}\sum_{r\in R^+} r$. Then we have

$$\begin{aligned}\theta_R(z;\tau) &:= \eta(\tau)^{n-|R^+|}\prod_{r\in R^+}\theta\left(\frac{(r,z)}{h};\tau\right)\\ &= \sum_{x\in w+W_{R,ev}} q^{\frac{(x,x)}{2h}}\sum_{g\in G_R}\mathrm{sn}(g)e\left(\frac{(gx,z)}{h}\right)\end{aligned}$$

for all $\tau \in \mathfrak{H}$ and $z \in \mathbb{C}\otimes W_R$. θ_R is in $J_{\frac{n}{2},\underline{R}}(\epsilon^{n+2N})$.

Some pictures



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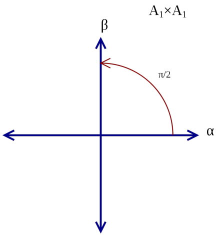

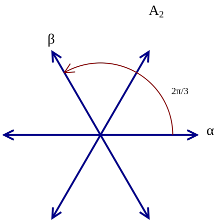

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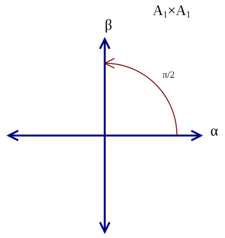

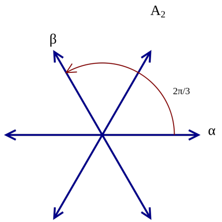

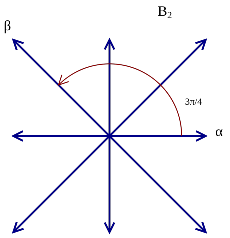

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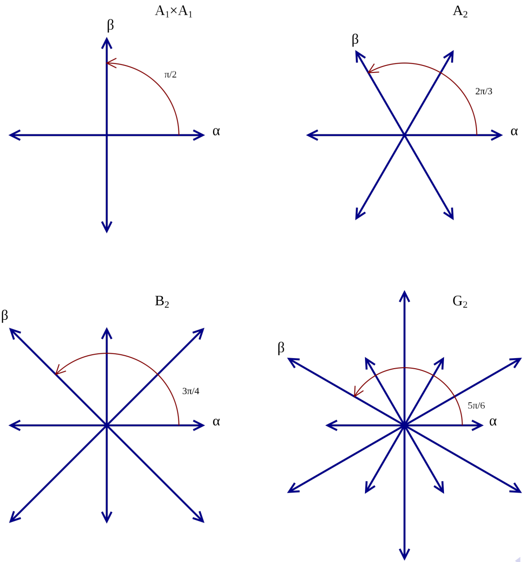

[Wikimedia User:Maksim]

Weight one theta blocks

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h	R	$\phi_R(z;\tau)$
2	$A_1 \oplus A_1$	$\vartheta^*(z_1)\vartheta^*(z_2)$
4	$A_1 \oplus A_1$	$\vartheta(z_1)\vartheta^*(z_2)$
6	$A_1 \oplus A_1$	$\vartheta(z_1)\vartheta(z_2)$
8	A_2	$\eta^{-1}\vartheta(z_1)\vartheta(z_2)\vartheta(z_1+z_2)$
10	B_2	$\eta^{-2}\vartheta(z_1)\vartheta(z_2)\vartheta(z_1+z_2)\vartheta(z_1+2z_2)$
14	G_2	$\eta^{-4}\vartheta(z_1)\vartheta(z_2)\vartheta(z_1+z_2)\vartheta(2z_1+z_2)\vartheta(3z_1+z_2)\vartheta(3z_1+2z_2)$

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We can choose

$$\begin{aligned} B_2^+ &= \{(1,-1),(0,1),(1,0),(1,1)\} \\ &= \{r_1, r_2, r_3 = r_1 + r_2, r_4 = r_1 + 2r_2\}, \\ F_{B_2} &= \{r_1, r_2\}. \end{aligned}$$

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A calculation shows $G_{B_2} = \{\pm Id, \pm s_{r_1} s_{r_2}, \pm s_{r_1}, \pm s_{r_2}\} \cong D_4$ with

$$\mathrm{sn}(\pm Id) = \mathrm{sn}(\pm s_{r_1} s_{r_2}) = 1, \quad \mathrm{sn}(\pm s_{r_1}) = \mathrm{sn}(\pm s_{r_2}) = -1.$$

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We find $h = 3$ and $w = \left(\frac{3}{2}, \frac{1}{2}\right)$.

An example: $R = B_2$

$$\begin{aligned} W_{B_2} &= \left\{ x \in \mathbb{R}^2 : \frac{(x, r)}{3} \in \mathbb{Z} \ \forall r \in B_2 \right\} \\ &= \{ x = (x_1, x_2) \in \mathbb{Z}^2 : x_1 \equiv x_2 \equiv 0 \pmod 3 \} \end{aligned}$$

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$$W_{B_2,ev} = \left\{ x \in W_{B_2} : \frac{(x,x)}{6} \in \mathbb{Z} \right\}$$
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$x = (x_1, x_2) = x_1 r_1 + (x_1 + x_2) r_2$, compute action on simple roots:

$$\pm Id(x) = \pm \left[x_1 r_1 + (x_1 + x_2) r_2 \right],$$
$$\pm s_{r_1} s_{r_2}(x) = \pm \left[-x_2 r_1 + (x_1 - x_2) r_2 \right],$$
$$\pm s_{r_1}(x) = \pm \left[x_2 r_1 + (x_1 + x_2) r_2 \right],$$
$$\pm s_{r_2}(x) = \pm \left[x_1 r_1 + (x_1 - x_2) r_2 \right].$$

An example: $R = B_2$

We make the changes of variable $z_1 = \frac{(r_1,z)}{3}$ and $z_2 = \frac{(r_2,z)}{3}$ to obtain

$$\theta_{B_2}(z;\tau) = \eta(\tau)^{-2}\theta(z_1;\tau)\theta(z_2;\tau)\theta(z_1+z_2;\tau)\theta(z_1+2z_2;\tau)$$

$$= \sum_{\substack{x\in\left(\frac{3}{2},\frac{1}{2}\right)+\mathbb{Z}^2 \\ x_1\equiv x_2\equiv 0 \pmod 3 \\ x_1\equiv x_2 \pmod 2}} q^{\frac{x_1^2+x_2^2}{6}}$$

$$\times\left[\zeta_1^{x_1}\zeta_2^{x_1+x_2}+\zeta_1^{-x_1}\zeta_2^{-x_1-x_2}+\zeta_1^{-x_2}\zeta_2^{x_1-x_2}+\zeta_1^{x_2}\zeta_2^{-x_1+x_2}\right.$$

$$\left.-\zeta_1^{x_2}\zeta_2^{x_1+x_2}-\zeta_1^{-x_2}\zeta_2^{-x_1-x_2}-\zeta_1^{x_1}\zeta_2^{x_1-x_2}-\zeta_1^{-x_1}\zeta_2^{-x_1+x_2}\right].$$

An application to colored partitions

Let

$$\mathcal{C}_k(a_1 z, a_2 z, \ldots, a_{\frac{k+\delta_{\text{odd}}(k)}{2}} z; \tau) := \mathcal{C}(0;\tau)^{\frac{k-\delta_{\text{odd}}(k)}{2}} \prod_{i=1}^{\frac{k+\delta_{\text{odd}}(k)}{2}} \mathcal{C}(a_i z;\tau),$$

where $\mathcal{C}(z;\tau) := \prod_{n\geq 1} \frac{1-q^n}{(1-\zeta q^n)(1-\zeta^{-1}q^n)}$

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$$\mathcal{C}_k(z;\tau) := \mathcal{C}_k(kz, (k-2)z, \ldots, (2-\delta_{\mathrm{odd}}(k))z;\tau)$$
$$= \prod_{n\geq 1} \frac{1-\delta_{\mathrm{odd}}(k)q^n}{(1-\zeta^{\pm k}q^n)(1-\zeta^{\pm(k-2)}q^n)\cdots(1-\zeta^{\pm(2-\delta_{\mathrm{odd}}(k))}q^n)}.$$

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Notice that

$$C_k(0;\tau) = P_k(\tau) := \sum_{n\geq 0} p_k(n)q^n = \prod_{n\geq 1} \frac{1}{(1-q^n)^k}.$$

An example of the above theorem

Let $\Phi_\ell(\zeta)$ denote the ℓ-th cyclotomic polynomial.

Theorem (R.-Tripp-W)

Suppose $k \equiv -10 \pmod{\ell}$ *for a prime* $\ell \equiv 3 \pmod{4}$. *Then for* $n \geq 0$ *we have the divisibility relation*

$$\Phi_\ell(\zeta) \,\Bigg|\, \left[q^{\ell n + 5\frac{\ell^2-1}{12}}\right] \mathcal{C}_k(z;\tau).$$

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Corollary

Suppose $k \equiv -10 \pmod{\ell}$ *for a prime* $\ell \equiv 3 \pmod{4}$*. Then we have the Ramanujan-type congruence*

$$p_k\left(\ell n + 5\frac{\ell^2-1}{12}\right) \equiv 0 \pmod{\ell}.$$

Proof

- The discussion of θ_{B_2} shows that

$$\prod_{n\geq 1}(1-q^n)^2(1-\zeta_1^{\pm 1}q^n)(1-\zeta_2^{\pm 1}q^n)(1-(\zeta_1\zeta_2)^{\pm 1}q^n)(1-(\zeta_1\zeta_2^2)^{\pm 1}q^n)$$

vanishes at the coefficient $[q^{\ell n+5\frac{\ell^2-1}{12}}]$ when ζ_1 and ζ_2 are set to ℓ-th roots of unity.

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- Set $z_1 = 4z$ and $z_2 = 2z$ then multiply the numerator and denominator of $C_k(z;\tau)$ by the above product.

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- The denominator factors into terms of the form $1 - q^{\ell n} + \Phi_\ell(\zeta) \cdot f(z;\tau)$ for some function f.

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- The denominator factors into terms of the form $1 - q^{\ell n} + \Phi_\ell(\zeta) \cdot f(z;\tau)$ for some function f.
- The numerator is the above product which we have shown is divisible by $\Phi_\ell(\zeta)$ on the q-exponents we're concerned with.

Stanton-type Conjectures for k-colored partitions

Theorem (Bringmann-Gomez-R.-Tripp, 2021)

There are families of Stanton-type conjectures that appear to hold for these families.

Numerical Examples

Crank	Unimodal?
$\mathcal{C}_3(2,1;z;\tau)$	$\forall n > 7$
$\mathcal{C}_3(3,1;z;\tau)$	no
$\mathcal{C}_3(3,2;z;\tau)$	$\forall n > 6$

(A) $k = 3$

Crank	Unimodal?
$\mathcal{C}_4(2,1;z;\tau)$	$\forall n > 1$
$\mathcal{C}_4(3,1;z;\tau)$	no
$\mathcal{C}_4(4,1;z;\tau)$	no
$\mathcal{C}_4(3,2;z;\tau)$	$\forall n > 1$
$\mathcal{C}_4(4,2;z;\tau)$	no
$\mathcal{C}_4(4,3;z;\tau)$	$\forall n > 23$

(B) $k = 4$

Crank	Unimodal?
$\mathcal{C}_5(3,2,1;z;\tau)$	$\forall n > 9$
$\mathcal{C}_5(4,2,1;z;\tau)$	no
$\mathcal{C}_5(5,2,1;z;\tau)$	no
$\mathcal{C}_5(4,3,1;z;\tau)$	$\forall n > 11$
$\mathcal{C}_5(5,3,1;z;\tau)$	no
$\mathcal{C}_5(5,4,1;z;\tau)$	$\forall n > 9$
$\mathcal{C}_5(4,3,2;z;\tau)$	$\forall n > 10$
$\mathcal{C}_5(5,3,2;z;\tau)$	no
$\mathcal{C}_5(5,4,2;z;\tau)$	$\forall n > 13$
$\mathcal{C}_5(5,4,3;z;\tau)$	$\forall n > 13$

(C) $k = 5$

Crank	Unimodal?
$\mathcal{C}_6(3,2,1;z;\tau)$	$\forall n > 1$
$\mathcal{C}_6(4,2,1;z;\tau)$	no
$\mathcal{C}_6(5,2,1;z;\tau)$	no
$\mathcal{C}_6(6,2,1;z;\tau)$	no
$\mathcal{C}_6(4,3,1;z;\tau)$	$\forall n > 5$
$\mathcal{C}_6(5,3,1;z;\tau)$	no
$\mathcal{C}_6(6,3,1;z;\tau)$	no
$\mathcal{C}_6(5,4,1;z;\tau)$	$\forall n > 11$
$\mathcal{C}_6(6,4,1;z;\tau)$	no
$\mathcal{C}_6(6,5,1;z;\tau)$	$\forall n > 21$
$\mathcal{C}_6(4,3,2;z;\tau)$	$\forall n > 14$
$\mathcal{C}_6(5,3,2;z;\tau)$	no
$\mathcal{C}_6(6,3,2;z;\tau)$	no
$\mathcal{C}_6(5,4,2;z;\tau)$	$\forall n > 19$
$\mathcal{C}_6(6,4,2;z;\tau)$	no
$\mathcal{C}_6(6,5,2;z;\tau)$	$\forall n > 20$
$\mathcal{C}_6(5,4,3;z;\tau)$	$\forall n > 7$
$\mathcal{C}_6(6,4,3;z;\tau)$	no
$\mathcal{C}_6(6,5,3;z;\tau)$	$\forall n > 32$
$\mathcal{C}_6(6,5,4;z;\tau)$	$\forall n > 19$

(D) $k = 6$

TABLE 1. Cranks for the given value of k

Conjecture 4.1. *Let* $\mathcal{D}(z;\tau) := \mathcal{C}_k(a_1, a_2, \ldots, a_{\frac{k+\delta_{2\nmid k}}{2}}; z; \tau)$ *for some* $a_1 > a_2 > \cdots > a_{\frac{k+\delta_{2\nmid k}}{2}} > 0$ *and* $k \geq 3$. *Then* $\mathcal{D}(z;\tau)$ *is eventually unimodal if and only if* $a_1 - a_2 = 1$.

Thank you!!!