Recent problems in partitions and other combinatorial functions

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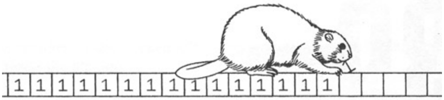
Oregon State University Math Colloquium
[Credit: Math∩Programming Blog]



Partitions

Integer partitions

Definition

An **integer partition** *of n is a sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ such that*

$$\lambda_1 + \ldots + \lambda_k = n.$$

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$$13 = \textit{Eleven Plus Two} = \textit{Twelve Plus One}.$$

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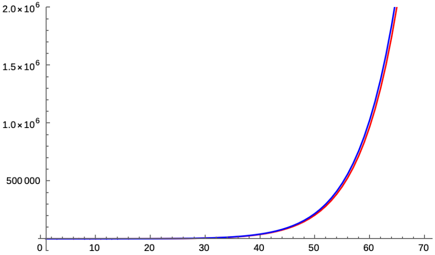
[Credit:Jon Perry (June 2011)]



Even more precise

- This can be extended to an **exact formula** of Rademacher:

$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{k\geq 1} \frac{A_k(n)}{k} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right).$$

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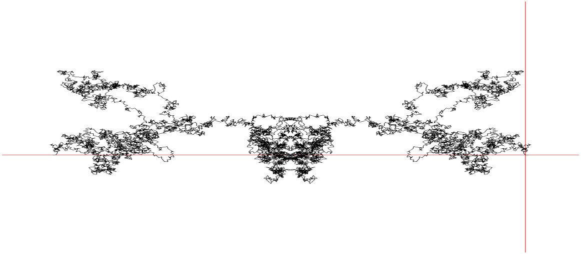

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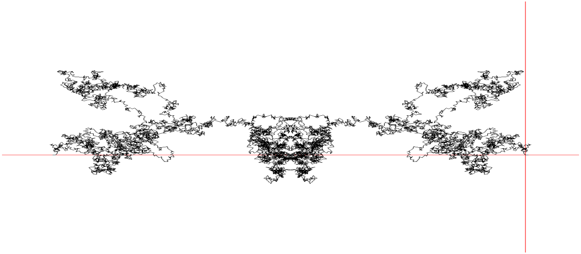

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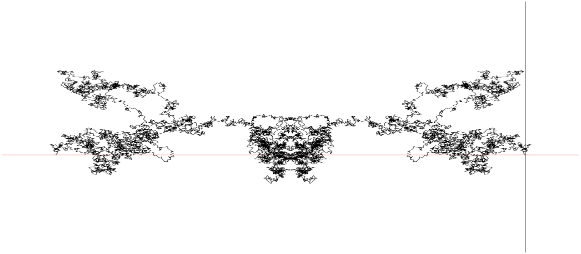

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$$p(n)^2 \geq p(n-1)p(n+1) \qquad (n > 25).$$

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Theorem (Ramanujan's Congruences 1919; Hardy-Ramanujan)

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- Originally only q-series (analytic) proofs (gen. functions).

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λ=5+4+3+1
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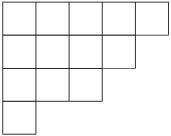

$\longrightarrow$ λ'=4+3+3+2+1
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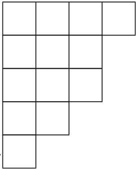

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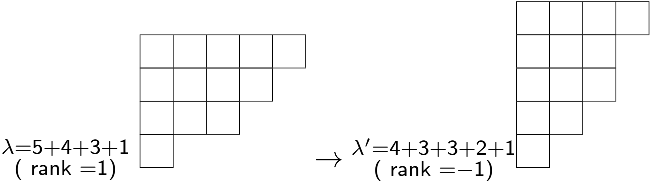

- $N(m, n) := \#\{\text{ptns of } n \text{ with rank } m\}$,
  $N(m, q; n) := \#\{\text{ptns of } n \text{ with rank } \equiv m \pmod{q}\}$.

Dyson's Conjecture

Theorem (Conjecture of Dyson 1944, proven by Atkin and Swinnerton-Dyer in 1954)

We have

$$N(0,5;5n+4) = N(1,5;5n+4) = \ldots = N(4,5;5n+4).$$

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- The generating function is ($q := e^{2\pi i\tau}$, $\tau \in \mathbb{H}$, $\zeta := e^{2\pi iz}$, $z \in \mathbb{C}$, $(a;q)_n = (a)_n := \prod_{j=0}^{n-1}(1-aq^j)$,):

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$$R(\zeta;q) := \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} N(m,n)\zeta^m q^n = \sum_{n\geq 0} \frac{q^{n^2}}{(\zeta q)_n(\zeta^{-1}q)_n}.$$

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- For z any other *torsion point* $z \in \mathbb{Q}\tau + \mathbb{Q}$, this gives a key example of a "mock modular form."
- This helped spur the growth of the field of mock modular forms/harmonic Maass forms and led to many new applications in analytic and arithmetic properties of partitions.

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Cranks "explain" Ramanujan's congruences mod 5, 7, *and* 11.

Remarks on cranks

Generating Function

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- This works out since $\vartheta'(0;\tau) = -2\pi\eta^3(\tau)$.

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Here, Φ_ℓ is the ℓ-th cyclotomic polynomial, and divisibility is as Laurent polynomials.

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- Looking at the coefficients of powers of q is uncommon.

A question of Stanton

Question (Stanton)

Ranks and cranks distribute partitions into equinumerous sets. Can we find a **direct bijection***?*

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```
.                                   1;
.                                1, 0, 0;
.                             1, 0, 0, 0, 1;
.                          1, 0, 0, 1, 0, 0, 1;
.                       1, 0, 1, 0, 1, 0, 1, 0, 1;
.                    1, 0, 1, 0, 1, 1, 1, 0, 1, 0, 1;
.                 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1;
.              1, 0, 1, 1, 1, 1, 2, 1, 2, 1, 1, 1, 1, 0, 1;
.  1, 0, 1, 1, 2, 1, 2, 2, 2, 2, 2, 1, 2, 1, 1, 0, 1;
1, 0, 1, 1, 2, 1, 3, 2, 3, 2, 3, 2, 3, 1, 2, 1, 1, 0, 1;
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Stanton's Conjecture

Definition (Stanton)

The **modified rank and crank** *are:*

$$rank^{*}_{\ell,n}(\zeta) := rank_{\ell n+\beta} + \zeta^{\ell n+\beta-2} - \zeta^{\ell n+\beta-1} + \zeta^{2-\ell n-\beta} - \zeta^{1-\ell n-\beta},$$

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where $\beta := \ell - \frac{\ell^2-1}{24}$.

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Conjecture (Stanton)

All of the following are Laurent polynomials with positive coefficients:

$$\frac{rank^*_{5,n}(\zeta)}{\Phi_5(\zeta)}, \quad \frac{rank^*_{7,n}(\zeta)}{\Phi_7(\zeta)}, \quad \frac{crank^*_{5,n}(\zeta)}{\Phi_5(\zeta)}, \quad \frac{crank^*_{7,n}(\zeta)}{\Phi_7(\zeta)}, \quad \frac{crank^*_{11,n}(\zeta)}{\Phi_{11}(\zeta)}.$$

Result for cranks

Theorem (Bringmann, Gomez, R., Tripp, 2021)

The crank part of Stanton's Conjecture is true.

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- It turns out that this relates to inequalities of crank numbers. . . more on such inequalities later.

Natural questions

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What do the positive coefficients mean?

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Is there a more general phenomenon?

A test case

Definition

The k-colored partitions are defined via generating functions as

$$\sum_{n\geq 0} p_k(n)q^n =: (q)_\infty^{-k}.$$

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  4. When $k \equiv -4, -6, -8, -10, -14, -26 \pmod{\ell}$:

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Question

Are there combinatorial interpretations for these congruences?

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Remark

The cases $k =\equiv -4, -6, -8, -10, -14 \pmod{\ell}$ can be seen as coming from Macdonald identities. The case of 26 is still a mystery (old question of Dyson, Serre, et al on η^{26}.

A tool for "discovering" crank functions

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Theorem (Tripp-R.-Wagner 2020)

There is an infinite family of crank functions $C_k(z;\tau)$which explain "most" congruences of colored partitions.

Examples

- Sample excluded case: There is an odd prime $p \equiv 2 \pmod 3$ with $p|(k+14)$, $\ell \equiv 2 \pmod 3$, and $\ell|(k+8)$.

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- The definition of the vectors is given for odd k by:

$$C_k(z;\tau) := \begin{cases} C_k(k,(k-2),1\ldots;\tau) & \nexists \ell = 3r+2|(k+14), \\ C_k(k+2,k-2,\ldots,1;\tau), & \text{otherwise.} \end{cases}$$

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- Idea: Use new theory of Gritsenko-Skoruppa-Zagier's *theta blocks* to give convenient constructions using Lie-theoretic formulas that make it easier to "discover" such functions in large families.

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Theorem (Bringmann-Gomez-R.-Tripp, 2021)

There are families of Stanton-type conjectures that appear to hold for these families.

Powers of the eta function

- Nekrasov-Okounkov formula ($\mathcal{P}$ is the set of partitions; $|\lambda|$ is the number partitioned)

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- $\mathcal{H}(\lambda)$ are the "hook lengths."
- First studied in the context of supersymmetric gauge theory.

Partition Inequalities

Theorem (De Salvo-Pak, Nicolas)

The partition function is eventually log-concave:

$$p(n)^2 \geq p(n-1)p(n+1), \quad (n > 25).$$

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- Another multiplicative inequality was given by Bessenrodt-Ono:

$$p(a)p(b) \geq p(a+b), \quad (a, b \geq 2 \ a+b > 8).$$

Conjectures for arbitrary eta function powers

Conjecture (Chern-Fu-Tang)

For $n, \ell \in \mathbb{N}$, $k \in \mathbb{N}_{\geq 2}$, $n > \ell$, $(k, n, \ell) \neq (2, 6, 4)$, *we have*

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Conjecture (Heim-Neuhauser)

The same holds for any $k \in \mathbb{R}_{\geq 2}$.

Final result

Theorem (Bringmann-Kane-R.-Tripp)

The conjecture of Chern-Fu-Tang is true.

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Theorem (Bringmann-Kane-R.-Tripp)

The conjecture of Chern-Fu-Tang is true.

- The proof uses exact formulas for $p_k(n)$ due to Iskander-Jain-Talvola. Then explicit error bounds and estimations, plus a big computer check of finitely many cases.

Thank you!!!