Periodicities for Taylor coefficients of half-integral weight modular forms

Larry Rolen (joint with Michael Mertens and Pavel Guerzhoy)

Vanderbilt University

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- This gives the connection to classical applications, e.g., divisor sums, representations of integers by quad. forms, partitions.
- ∞ is a "natural" point to expand near:
- The modular curves $\Gamma_0(N)\backslash\mathbb{H}$ aren't compact, so one has to add in the cusps, which are distinguished points.

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- Natural way to build these spaces: Poincaré series.
- Petersson slash action: $f|_k\gamma := (c\tau + d)^{-k} f(\tau)$.
- f is "modular" at $\gamma \iff f|_k\gamma = f$.

Poincaré series

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- Key fact (Petersson):

$$f \in S_k(N) \implies \langle f, P_{N,k,m} \rangle \doteq [q^m] f(\tau).$$

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- Sometimes they are necessary, e.g., in non-congruence subgroups when there are no cusps, Voight and Willis have studied elliptic expansions.
- They can still have "nice" congruences, for example as studied by Atkin and Swinnerton-Dyer/Winnie Li and Ling Long.

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- The Poincaré series in this case give very important functions.

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- Leads to locally harmonic Maass forms (Bringmann-Kane-Kohnen), applications to L-values and Tunnell's Theorem (Ehlen-Guerzhoy-Kane-R.).

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- Zagier: Traces of singular moduli

Normalizing elliptic expansions

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- This leads to the expansion

$$(1-w)^{-k} f\left(\frac{\tau_0-\overline{\tau_0}w}{1-w}\right) = \sum_{n\geq 0} \partial^n f(\tau_0)\frac{(4\pi y_0 w)^n}{n!} \quad (|w|<1),$$

where $\partial_k = \partial := \frac{1}{2\pi i}\frac{d}{d\tau} - \frac{k}{4\pi y}$.

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- Special cases: Rodriguez-Villegas/Zagier: Some Taylor coefficients of Eisenstein series are Hecke L-values.
- $\rightsquigarrow$ Computable criterion to check if $p \equiv 1 \pmod 9$ is a sum of two rational cubes.
- Other than Eisenstein series, one of the "nicest" next examples is Jacobi's theta function $\vartheta_3(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}$.

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- Renormalizing gives an integer sequence:

$$(1-w)^{-\frac{1}{2}}\vartheta_3\left(\frac{i+wi}{1-w}\right) =: \vartheta_3(i)\sum_{n\geq 0}\frac{d(n)}{(2n)!}(\Phi w)^{2n}.$$

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- First few values of $d(n)$ are $1, 1, -2, 51, 849, -26199, \ldots$.

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Conjecture (Romik)

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Question

Is there a natural framework?

Our results

Theorem (Guerzhoy-Mertens-R.)

Let $f \in M_{k-\frac{1}{2}}(\Gamma_1(4N))$ have algebraic integral Fourier coefficients. Suppose $p > 3$ is split in $\mathbb{Q}(\tau_0)$ for a CM point τ_0.

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$$n_1 \equiv n_2 \pmod{(p-1)p^A} \implies$$

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Corollary

The conjecture of Romik is true.

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- It seems a more general version is also true in the inert prime case, but new techniques are required to cover higher levels.
- Recently, Wakhare and Wakhare-Vignat have also studied generalizations and refinements of Romik's conjecture.

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- In general, we get an isomorphism

$$\oplus_k \widetilde{M}_k(\Gamma) \cong \oplus_k M_k^*(\Gamma),$$

with $D := \frac{1}{2\pi i}\frac{d}{d\tau}$ acting on quasimodular forms, ∂ on almost holomorphic forms.

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- To pass to integral weight, we multiply by ϑ_3.
- Then work of Damerell and Katz on algebraicity of values must be interpreted correctly.

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- Thank you!