HARMONIC MAASS FORMS AND MOCK MODULAR FORMS: OVERVIEW AND APPLICATIONS

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We have seen in Michael Mertens' lectures that the "web of modularity" reverberates across many areas of mathematics and physics. In these lectures, we will discuss an extension of this web and its consequences and uses in different areas, and, in particular, in combinatorics and q-series. In fact, a lot of the best applications in the theory of harmonic Maass forms and motivating examples which historically motivated important results in our field have come from its interactions with these other subjects, in particular combinatorics and physics. For example, combinatorialists have a different set of tools and intuitions, and a knack for discovering interesting new examples of modular-type objects. Thus, collaboration and communication between the two areas, such as that established at this conference, is very fruitful in both directions.

Before we begin, I have two brief comments. Firstly, a survey of many of the things in these notes and more generally the theory and applications of harmonic Maass forms can be found in the book on harmonic Maass forms by Bringmann, Folsom, Ono, and myself. Secondly, if you have any comments (or questions) about any of these notes, such as typos or general questions about the material, proofs, how to apply the techniques discussed here, etc., please feel free to email them to me.

1. History and motivation

There are numerous beautiful examples of combinatorial generating functions which are both q-hypergeometric series and modular forms. As a fundamental example, consider the partition function $p(n)$, which counts the number of integer partitions of n, that is, the number of ways to write n as a sum of a sequence of non-increasing integers. As we have seen in the earlier lectures, the generating function $P(q) := \sum_n p(n)q^n$ is essentially a modular form of weight $-1/2$, and this can be used to prove congruences and estimates/exact formulas for the numbers $p(n)$. We saw this by using the classical product formula for P. There is another formula for P, which arises from a counting of partitions while keeping track of sizes of Durfee squares in the Ferrers diagrams. Namely,

$$P(q) = \sum_{n\geq 0} \frac{q^{n^2}}{(q)_n^2},$$

where $(a;q)_n = (a)_n := \prod_{j=0}^{n-1}(1-aq^j)$. This form is beautiful and useful, but hard to deduce modularity from. Being able to do so would be incredibly useful, and is a problem frequently encountered in this area, either explicitly or implicitly. This was summarized neatly in the following problem, popularized by George Andrews.

Problem (Andrews)**.** *How can one directly prove the modularity properties of $P(q)$ directly from the q-hypergeometric expression above, i.e., without first proving a $\sum = \prod$ identity?*

This problem is hard (in particular, unsolved). However, while modularity is difficult to pin down exactly on the level of q-hypergeometric series, modular forms do leave their fingerprints behind. This can be seen via the following elementary lemma, which follows directly from the transformations in the definition of a modular form. A proof of this lemma in this form can be found in Vlasenko and Zwegers' paper on Nahm's conjecture.

Lemma. *If $f \in M_k(\Gamma)$ for a subgroup Γ of finite index in $\mathrm{SL}_2(\mathbb{Z})$, then there are constants a, b such that as $\varepsilon \searrow 0$,*

$$e^{a/\varepsilon} f\left(\frac{i\varepsilon}{2\pi}\right) \sim be^{-k} + o(\varepsilon^N)$$

for all $N \geq 0$. That is, there is only one term in the asymptotic expansion **to all orders**.

The point is that we are letting the modular variable τ tend to 0 from above along the imaginary axis, or letting q tend to 1 radially from within the unit disk, and this can be related to the behavior at $i\infty$ by modularity, and the behavior there is easy to determine using the Fourier expansion.

However, among q-hypergeometric series, this type of asymptotic expansion is very rare. Almost all q-hypergeometric expansions have complicated expansions, usually of an infinite form, which are very far from this elementary asymptotic behavior. In fact, in his original letter where Ramanujan describes his mock theta functions, he begins by giving a "generic" example with an infinite type expansion which doesn't look related to modular forms.

Example 1. *A famous conjecture of Nahm claims that certain q-hypergeometric series are modular if and only if certain elements of the so-called Bloch group (an object from K-theory) are torsion. The first case of this conjecture claims to predict exactly those $A, B, C \in \mathbb{Q}$ for which*

$$F_{A,B,C}(q) := \sum_{n \geq 0} \frac{q^{\frac{An^2}{2} + Bn + C}}{(q)_n}$$

is modular. Zagier solved this by proving the following result.

Theorem 1 (Zagier)**.** *The series $F_{A,B,C}$ is modular for exactly 7 (explicitly given) triples $(A, B, C) \in \mathbb{Q}^3$.*

For example, these examples include the famous Rogers-Ramanujan function:

$$\sum_n \frac{q^{n^2}}{(q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$

where the right-hand "product-side" can be directly shown to be modular. How did Zagier prove this? Essentially, by combining the type of reasoning encoded in the elementary lemma above with a brilliant method for computing the asymptotic expansions near $q = 1$ of each of the functions $F_{A,B,C}$, he reduced the candidates for modularity to finitely many cases. In each of these cases, modularity is established by applying a $\sum = \prod$ identity, such as that for the Rogers-Ramanujan function above.

2. A surprising discovery

Ramanujan was of course well aware of examples such as, well, the Rogers-Ramanujan identities. Every example of such an identity, and correspondingly of a q-hypergeometric modular form, is highly interesting. As mentioned above, he computed (at least one) examples of q-hypergeometric asymptotic expansions near roots of unity like $q = 1$. Shortly before his untimely death, while isolated in India, he wrote a brief letter to Hardy describing a startling set of examples he discovered which he knew are not quite modular (in his language, are not "theta functions," or quotients, products, and linear combinations thereof), but which are so uncannily close that something must be very special about them. For instance, one of his functions was the series

$$f(q) := \sum_n \frac{q^{n^2}}{(-q)_n^2},$$

which looks similar to the expression for P above, but with a minus sign inserted. In particular, it doesn't satisfy the lemma above (which can be modified to approach other roots of unity, or, in τ, for τ approaching rational numbers vertically from above), but it doesn't have the kind of infinite, wild expansions which are typical for non-modular series. In particular, he considered the (up to a rational power of q) weight $1/2$ weakly holomorphic modular form

$$b(q) := (q;q^2)_\infty \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}.$$

Ramanujan then described a shocking numerical near cancellation between these two functions near $q=-1$, where they both explode. For example (using a modern computer we can compute that), $f(-0.998) \approx -6\cdot 10^{90}$. However,

$$f(q) + b(q) \approx 3.992 \approx 4.$$

N.B.: Where does the number 4 come from? We will address this question later.

This near miss example is what Ramanujan called a *mock theta function*, is it is "imitating" or pretending to be the "theta function" $b(q)$ near $q=-1$, and also has predictable behavior near other roots of unity. Ramanujan's letter goes on to give a definition of mock theta functions as he conceived of them, where he attempts to analytically describe the strangenesss of what he observed, and to give a number of further examples he claims satisfy similar properties. His examples gave tantalizing hints of a new structure of modular-type objects, and this feeling served as a final challenge of Ramanujan which tantalized number theorists for decades to come.

3. A framework for Ramanujan's mock theta functions

Building on earlier work expanding Ramanujan's examples and shedding light on their transformation properties, combinatorial implications, and various representations (such as those in a form compatible with Zwegers' three-fold path described below) and properties due to many people such as Andrews, Berndt, Hickerson, Watson, and a number of others, a framework for what sorts of functions Ramanujan was studying finally emerged about 80 years after his original letter, in the early 2000's. This can be explained using the theory of harmonic Maass forms, which was independently developed in a seminal paper due to Bruinier and Funke. There, they defined a harmonic Maass form, which can be roughly summarized as follows. Essentially, harmonic Maass forms are no longer required to be holomorphic, but rather to satisfy a second order differential equation instead (and, being in the kernel of a "Laplacian," they are termed "harmonic").

"Definition". *A harmonic Maass form (HMF) is a function which satisfies the following.*

(1) It transforms like a modular form.
(2) It is in the kernel of

$$\Delta_k := -\xi_{2-k}\xi_k,$$

where

$$\xi_k := 2i\,\mathrm{Im}(\tau)^k \overline{\frac{\partial}{\partial\overline{\tau}}}.$$

(3) It grows like at most as quickly as a weakly holomorphic modular form at the cusps (i.e., it has a principal part, and once this is subtracted, it decays rapidly).

The space of HMFs of weight k is denoted by H_k (with the subgroup of $\mathrm{SL}_2(\mathrm{Z})$ suppressed freely for convenience and clarity in these notes).

Note that classical modular forms satisfy the conditions of the definition of a HMF, as the operator ξ_k kills holomorphic functions by definition, and so Δ_k does too. The operator ξ_k plays a central role in the theory of harmonic Maass forms. It shuttles between "dual weights" k and $2-k$, which in many ways reflect dual worlds of existence. In particular, ξ_k applied to a HMF of weight k yields a modular object of weight $2-k$. Since Δ_k splits as a composition of two ξ_k operators (up to a sign, which is there for classical reasons), the image under ξ_k must be in the kernel of ξ_{2-k}. But this precisely means that the image is a holomorphic function, and so one must obtain a holomorphic modular form. By considering the growth condition at cusps, it turns out that one always obtains a *cusp form*. Thus, one obtains a map

$$\xi_k \colon H_k \to S_{2-k}.$$

Importantly, Bruinier and Funke used geometry to prove that this map is *surjective* (one can think of this as a sort of "Serre duality"), and, following Zagier, we commonly refer to the image of a HMF under it as the *shadow* of the HMF. Moreover, just as holomorphicity and translation invariance imply the existence of Fourier expansions of modular forms, harmonicity and translation invairance prove that HMFs have a Fourier expansion which splits into two pieces (since its a second order differential equation):

$$f \in H_k \implies f = f^+ + f^-.$$

Here, f^+ is called the *holomorphic part* and f^- is called the *non-holomorphic part*. Fittingly, f^+ is holomorphic, and f^- is not (assuming its non-zero, i.e., that we don't have a classical modular form). In particular, f^+ is an ordinary q-series, whose coefficients often encode some important object in whatever application we may be studying, and the coefficients of f^-, while decorated by certain non-holomorphic functions, are the same numbers which occur in the Fourier expansion of the shadow. Thus, the non-holomorphic part "comes from" classical modular forms and understanding it is equivalent to knowing about the shadow of f.

Definition 3.1. *A mock modular form is the holomoprhic part of a harmonic Maass form. A mock theta function is a mock modular form whose shadow is a unary theta series.*

Around the same time as Bruinier and Funke developed this theory, Zwegers proved that these same properties can be applied to modifications of Ramanujan's mock theta functions, and so, in this language, he proved the following in his famous Ph.D. thesis.

Theorem 2 (Zwegers)**.** *Ramanujan's mock theta functions are mock theta functions according to the definition above (up to multiplying by a rational power of q and a rescaling of q by a rational power).*

Thus, Zwegers proved that Ramanujan's functions are missing an extra piece, which, when added, "corrects" their modularity transformations. Moreover, this piece is "simpler" than the original mock theta function, as it contains the same information as a unary theta function, and so this completion is particularly useful for understanding the original mock theta function. This realization led to an explosion of applications across mathematics and physics, including to representation theory, combinatorics, black holes, and arithmetic geometry.

4. The three-fold path of Zwegers

Zwegers gave three related ways to realize Ramanujan's mock theta functions as examples of broad classes of functions. Firstly, he studied properties of *Appell-Lerch series*, which have a shape such as

$$\sum_{n\in\mathbb{Z}} \frac{q^{Q(n)}}{1-aq^n},$$

where the sum over $\mathbb{Z}$ can be replaced with other lattices, and $Q(n)$ can be a quadratic form, for instance. If you expand the denominator as a geometric series in a certain range, you can find expressions for *indefinite theta functions*. As an example of the shape of such an indefinite theta function, we have the following identity of Andrews for Ramanujan's fifth order mock theta function $f_0(q)$:

$$f_0(q) = \frac{1}{(q)_\infty}\left(\sum_{\substack{n+j\geq 0\\ n-j\geq 0}} - \sum_{\substack{n+j<0\\ n-j<0}}\right)(-1)^j q^{n(5n+1)/2-j^2}.$$

This is called an indefinite theta function as the quadratic form in the exponents of q is indefinite (has both positive and negative eigenvalues). As mentioned in the earlier lectures, this means that a sum over all choices of (n, j) would diverge, which is why we must do something more complicated such as summing only in a certain range.

N.B.: Much more on these functions, with the explanations of the ideas of the proofs and a discussion of very recent developments on indefinite theta functions and generalizations of harmonic Maass forms into a sequence of spaces of modular-type objects is discussed in detail (with exercises) in the notes on indefinite theta functions on the front page of my personal website. In accordance with the comment at the beginning of these notes, collaboration with combinatorics may play a key role in the development of these new structures.

Finally, Zwegers showed mock modularity properties of coefficients of meromorphic Jacobi forms. Important work of Dabholkar, Murthy, and Zagier follows up on this, and the reader (including readers with interest in q-series) are strongly encouraged to read their work as well. It should also be pointed out that coefficients of Jacobi forms, for example of infinite products in two variables, has often been important in combinatorics, for example, in using the constant term method. For instance, in his important memoir on generalized Frobenius partitions, Andrews proves formulas which can be used to connect combinatorial generating functions to both coefficients of Jacobi forms and mock modular forms, as they fit into the examples along the lines of what is briefly discussed here.

5. A few important applications

Here, we will survey just a few of the many topics where mock modular forms play a role.

5.1. **Combinatorics.** As a starting point, let us consider the Ramanujan congruences modulo 5 and 7 for $p(n)$:

$$p(5n+4) \equiv 0 \pmod 5, \qquad p(7n+5) \equiv 0 \pmod 7.$$

Dyson conjectured, and Atkin and Swinnerton-Dyer proved, that these congruences are combinatorially explained by the *rank* statistic of a partition π, $r(\pi)$, which is the largest part minus the number of parts. If we form the generating function of partitions refined to keep track of ranks:

$$\mathcal{R}(\zeta; q) := \sum_{n} \sum_{\pi \text{ is a partition of } n} q^n \zeta^{r(\pi)},$$

where $\zeta := e^{2\pi i z}$ and $z \in \mathbb{C}$, then we observe that $\mathcal{R}(1; q) = P(q)$ is modular, and $\mathcal{R}(-1; q) = f(q)$ is mock modular. Bringmann and Ono proved that these two observations are instances of the following fact.

Theorem 3 (Bringmann-Ono)**.** *For any root of unity ζ, $\mathcal{R}(\zeta; q)$ is a mock modular form (up to minor considerations such as multiplying by a rational power of q).*

This theorem immediately implies many congruences, and also allows one to study asymptotics of ranks very efficiently. We will see how these two properties, congruences and asymptotics, can be studied for mock modular forms in generality below.

5.2. **Arithmetic Geometry.** As an example of the applications to geometry, we consider the picture for elliptic curves. Suppose that E is an elliptic curve over $\mathbb{Q}$. Then by the Modularity Theorem, there is a weight 2 newform associated to the curve, which we will denote by f_E. We denote by Λ_E the lattice such that $E \cong_{\mathbb{C}} \mathbb{C}/\Lambda_E$. We will sketch an idea due to Guerzhoy, which allows us to give a "canonical" lifting under the (∞-to-1) map $\xi_0 \colon H_0 \to S_2$ (recall that Bruinier and Funke proved this is surjective). Crucially, this is compatible with "arithmetic," such as p-adic properties. Other methods for lifting, such as following Bruinier and Funke's proof or using the method of Poincaré series sketched below, either aren't as explicit (the following method relies only on classical modular forms theory and classical elliptic functions and is direct), or destroy arithmetic (writing a cusp form as a sum of cuspidal Poicnaré series is always possible in principle, but is not easy to understand explicitly and destroys the arithemetic properties by possibly expressing a q-series with integral coefficients using coefficients with transcendental coefficients; this method is better-suited to existence proofs and asymptotic analysis).

Consider the Weierstrass p function $\wp$, which is an elliptic function. The (negative of the) anti-derivative is the *Weierstrass zeta function*:

$$\wp(\Lambda_E; z) = -Z'(\Lambda_E; z).$$

Now Z is not elliptic, but only nearly so. Now, considering the modular form $f_E =: \sum_{n\geq 1} a_n q^n$ again, we consider the formal antiderivative (known as the *Eichler integral*)

$$\mathcal{E}_E(z) := \sum_{n\geq 1} \frac{a_n}{n} q^n,$$

so that

$$q\frac{d}{dq}\left(\mathcal{E}_E\right) = f_E.$$

Now this function is nearly modular of weight 0, but not quite. This is because slashing in weight 0 and 2 (and only in this case!) commutes with differentiation, in the sense the for any function g on the upper half-plane and any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

$$D\left(g|_0\gamma\right) = (Dg)|_2\gamma$$

where $D := q\frac{d}{dq}$. Thus, for any γ in the group Γ under which f_E transforms, the function $\mathcal{E}_E|_0(1-\gamma)$ has derivative equal to zero, and so is a constant. Thus, we obtain a map

$$\Psi\colon \Gamma \to \mathbb{C}$$

by setting $\Psi(\gamma)$ to be the constant obtained by evaluating $\mathcal{E}_E|_0(1-\gamma)$ at any point in $\mathbb{H}$. But classical theory of Eichler and Shimura says that the image of Ψ is exactly the lattice Λ_E! Thus, the "errors to modularity" of $\mathcal{E}_E$ all live in this lattice. Thus, if we consider the function $Z(\Lambda_E; \mathcal{E}_E)$, it would actually be modular of weight 0 if Z were elliptic, as errors to modularity would be swallowed up by lattice invariance. However, since Z is so very nearly an elliptic function, it can be corrected with a minor non-holmorphic correction term. Working this out, one finds that $Z(\Lambda_E; \mathcal{E}_E)$ is really a *mock modular form.* Moreover, the non-holomorphic part is especially simple, so computing the shadow of the mock modular form is essentially trivial, and easily returns the original cusp form f_E. Thus, we have answered our question of how to find explicit lifts under ξ_0. In a paper of Alfes, Griffin, Ono, and myself, we explored the implications for elliptic curves. By applying work of Bruinier and Ono, as well as Waldspurger and Kohnen-Zagier, it turns out that we can directly construct a single HMF whose holomorphic part encodes the vanishing of L-derivatives of quadratic twists of E, and whose non-holomorphic part has coefficients encoding the vanishing of the central L-values of the twists. By Birch and Swinnerton-Dyer conjecture, these coefficients should thus determine the rank of the quadratic twists of E up to rank 2, that is, they "know" whether the rank is 0, 1, or greater than 1.

More generally, these types of lifting problems are central in many questions on harmonic Maass forms, and as we now know, in classical modular forms. Returning to q-series, recall that *Lehmer's conjecture* states that all coefficients $\tau(n)$ of the Ramanujan Delta function $\Delta(\tau)$ are non-zero. Again by Bruinier-Funke, we can show that "nice" lifts of Δ to weight $2 - 12 = -10$ exist. It turns out that Lehmer's conjecture is equivalent to the coefficients of the corresponding mock modular form being transcendental. Thus, an explicit understanding of the lifting problem in a similar manner which is compatible with arithmetic for higher weights would be highly interesting.

6. Quantum Modular forms

Following Zagier, a *quantum modular form* of weight k, roughly speaking, is a function $f\colon \mathbb{Q}\cup\infty \to \mathbb{C}$ such that the "errors to modularity" $f|k(1-\gamma)$ are "nice functions" for all γ in some subgroup of $\mathrm{SL}_2(\mathbb{Z})$ (as this sounds similar to the previous section, it is worth pointing out that Eichler integrals can be used to define quantum modular forms). What nice means depends on the context, and the point is that this definition includes many different examples with a different flavor which Zagier surveyed in his original paper on the subject and which have been developed since. A typical behavior might be a function which is defined on $\mathbb{Q}$, with no way or no obvious way to extend it to $\mathbb{R}$, but whose cocycles become not only defined on $\mathbb{R}$ but also become $\mathcal{C}^{10}, \mathcal{C}^{\infty}$, analytic, etc.

As one example, recall that above we saw that Ramanujan's mock theta function $f(q)$ is almost cancelled out by the (essentially) modular form $b(q)$ near $q=-1$:

$$f(q)+b(q)\approx 4.$$

Ramanujan conjectured this value 4 in his letter, as well as similar almost cancelling values of $f(q)\pm b(q)$ at other roots of unity. It turns out that these constants which Ramanujan studied encode a hidden structure, namely, a quantum modular form. This is a good example of the fact that going back to the original sources in a subject and really reading them closely can be very useful in any field. The hidden structure behind Ramanujan's calculations is revealed in the following elegant result of Folsom, Ono, and Rhoades.

Theorem 4 (Folsom-Ono-Rhoades). *For every even order $2k$ root of unity ζ, we have the following radial limit formula:*

$$\lim_{q\to\zeta}\big(f(q)-(-1)^k b(q)\big) = -4\sum_n (1+\zeta)^2(1+\zeta^2)^2\cdots(1+\zeta^n)^2\zeta^{n+1}.$$

Note the right hand side is actually a *finite sum* for any root of unity ζ. As an exercise, consider what happens for odd order roots of unity. What is the meaning of the right hand side? We already know that $f(q)$ is a specialization of the rank generating function $\mathcal{R}$. It turns out that there is an elegant radial limit theorem for any other specialization of $\mathcal{R}$, which are mock modular as discussed above. To explain this, we need the it crank generating function $\mathcal{C}(\zeta;q)$, which si defined in the same way as $\mathcal{R}$ but with "rank" replaced by "crank," another partition statistic which simultaneously explains Ramanujan's congruences modulo 5, 7, *and* 11. This was conjectured by Dyson and found by Andrews and Garvan. The generating function enjoys the infinite product formula:

$$\mathcal{C} = \frac{(q)_\infty}{(\zeta q)_\infty(\zeta^{-1}q)_\infty}.$$

This is essentially the inverse of the Jacobi theta function times η^2, and so it is a meromorphic Jacobi form of weight 1/2. Thus, as discussed in the previous lectures on modular forms, its specializations, including $b(q)$, are all weakly holomorphic modular forms of weight 1/2. Furthermore, if we consider a generating function $\mathcal{U}(\zeta;q)$ which encodes strongly unimodal sequences by size and a statistic on them which is also called rank, then it turns out that

$$\mathcal{U}(\zeta;q)=\sum_n(-\zeta q)_n(-\zeta^{-1}q)_n\cdot q^{n+1}.$$

Folsom, Ono, and Rhoades then proved the following general limiting formula, which pleasingly connects three types of combinatorial generating functions in one and generalizes the above formula for $f(q)$ and $b(q)$.

Theorem 5 (Folsom-Ono-Rhoades). *Under certain conditions on the roots of unity ζ,ζ', and for certain elementary factors $*$, we have that*

$$\lim_{q\to\zeta'}\left(\mathcal{R}(\zeta;q) - *\mathcal{C}(\zeta;q)\right) = \mathcal{U}(-\zeta;\zeta').$$

This can be proved using asymptotic calculations, which require significant work. The reader is also referred to later, simpler proofs for such formulas by Zudilin, and later by Folsom, Ono, and Rhoades, and I highly recommend these works especially for a combinatorial audience. Since specializations of $\mathcal{R}$ are mock, and specializations of $\mathcal{C}$ are modular, it is natural to ask what kind of function $\mathcal{U}$ is. We noted above that it becomes a finite sum for roots of unity ζ, so it is well defined as a function on $z \in \mathbb{Q}$. As an example, $\mathcal{U}(-1;q) = \sum_n (q)_n^2 q^{n+1}$, Byrson, Pitman, Ono, and Rhoades proved that this function is actually (at roots of unity)

$$\mathcal{U}(-1;q) = F(q^{-1}),$$

where $F(q)$ is Kontsevich's function

$$F(q) = \sum_n (q)_n.$$

This function is known to be a quantum modular form, as Zagier proved by establishing his "strange identity"

$$F(q) \text{``}=\text{''} -\frac{1}{2} \sum_{n \geq 1} n\chi_{12}(n) q^{\frac{n^2-1}{24}}.$$

Quotation marks are used here as the left hand side now really only makes sense at roots of unity, while the *partial theta function* (it is summed over only half of a lattice) is only defined for $|q| < 1$. Note that this expansion is very closely related to the theta function representation for $\eta(\tau)$. The world where $F(q)$ lives, the *Habiro ring* which is known to encode invariants in knot theory, can be thought of as a set of "analytic functions at the roots of unity." In particular, we can consider the expansion of this function around roots of unity, such as the following one around $q = 1$:

$$\sum_n (1-q;1-q)_n =: \sum_n \xi_n q^n.$$

It is an instructive exercise to prove that the coefficients on the right hand side converge, as well as that similar expansions make sense at other roots of unity. These coefficients ξ_n are important numbers in knot theory. They were used by Zagier in the study of *Vassiliev invariants.* Just as modular and mock modular (as we will see) forms are very useful in studying asymptotics, Zagier used quantum modularity (in current language) to study the growth properties of these numbers. It also turns out that the same numbers count a number of interesting things in combinatorics, and these were further studied by Andrews and Sellers. They observed and proved that they satisfy infinitely many Ramanujan-type congruences. Further work by Guerzhoy, Kent, and myself explained this from a general quantum modular perspective and for general types of partial theta functions. Thus, in contrast to Ramanujan's statement that "false" (essentially the same theory as partial) theta functions "do not enter mathematics as beautifully" as his mock theta functions, they do enter into mathematics, in fact Ramanujan's mathematics, quite beautifully. Though, to be fair, it is difficult for most things to match the beauty of the mock theta functions.

7. General philosophy of applications to combinatorics

We conclude with a general discussion of how the theory of harmonic Maass forms gives powerful tools for studying two types of properties of sequences of numbers when you are lucky enough to have mock modular forms lurking: congruences and asymptotics.

7.1. **Congruences.** We have seen that modular forms are a powerful tool for proving congruences of arithmetic sequences. Here, we briefly sketch an approach for obtaining congruences for mock modular forms. The key idea is to reduce to the classical modular case. For instance, if the shadow is a linear combination of unary theta functions (for example, for a mock theta function; this property is also almost required for most combinatorial sequences as mock modular forms don't usually have integral Fourier coefficients otherwise). Then we are lucky, and we can twist away the shadow. As we saw in the earlier lectures, sieving operators, which restrict the Fourier coefficients of a modular form by throwing away all that don't lie in a fixed arithmetic progression, preserve modularity. The proof of that fact only used the transformation properties, i.e., it is a statement on the level of slash operators. Thus, sieving the coefficients of a HMF also yields a new HMF. But if the shadow is supported on finitely many square classes, then we can choose a progression where the coefficients of the shadow vanish. This then reduces us to the world of classical modular forms, and we can generically check in many cases that this procedure doesn't also kill the holomorphic part. By combining with the classical theory of modular form congruences, we thus obtain many congruences for coefficients of the mock modular form we may want to study. For instance, this already tells us, by applying Treneer's thesis, that mock theta functions satisfy infinitely many linear congruences. A number of papers have applied this technique to obtain strong results.

7.2. **Asymptotics.** We have seen that modularity can be used to prove asymptotics, and indeed exact formulas, for the partition function $p(n)$ by means of a Tabuerian Theorem (for the asymptotic) or the circle method (for more precision, depending on which variant of the circle method is used). Rademacher's formula turns out to be a hint of a more general structure of modular forms. We want a procedure to take any modular form (or mock modular form; as we shall see, answering the first question naturally leads us to the second) and give a procedure for computing the asymptotics of the coefficients of the modular form quickly. In the case of the partition function, we saw that the circle method consists of a careful study of the generating function near its singularities, and so its principal parts at (all equivalent in this case) the cusps dictate the asymptotic growth properties. The theory of harmonic Maass forms gives a way to diagonalize spaces based on principal parts. As is well-known, for general congruence subgroups, weakly holomorphic modular forms with arbitrary principal parts don't exist. However, *harmonic Maass* forms are allowed to have arbitrary principal part. Considering just the cusp ∞ for simplicity, for any $k \leq 0$, any $m \geq 1$, and any congruence subgroup Γ, we have a *Maass-Poincaré series* $F_{k,m,\Gamma} \in H_k(\Gamma)$ such that

$$F_{k,m,\Gamma} = q^{-m} + O(1)$$

at ∞, and $F_{k,m,\Gamma}$ is holomorphic at all inequivalent cusps. This series is a lift of the classical Poincaré cusp forms under ξ_k, and it is defined by a similar procedure as we saw for the Eisenstein series and other Poincaré series before; namely, by averaging a certain function hit with slash operators to force modularity while retaining good convergence and growth properties. Since there are no (non-constant) holomorphic modular forms of weight $k \leq 0$, this means that we can take the appropriate linear combination of these Poincaré series which matches the principal part of any weakly holomorphic, and more generally any mock, modular form in these weights, and the difference is simply a constant. The advantage here is that the series $F_{k,m,\Gamma}$ have completely explicit formulas for their Fourier coefficients, in terms of Kloosterman sums and Bessel functions, just as Rademacher found for $p(n)$. Thus, we can quickly determine an *exact formula* for the coefficients of any of these forms with a routine, finite check. We conclude by briefly considering other weights. For weights between 0 and 2, convergence of the Poincaré series is an issue. However, general work of Goldfeld and Sarnak can be used to analytically continue the Poincaré series in weights $1/2$ and $3/2$ (which, as described above, are often the ones of combinatorial interest), and the same formulas for Fourier expansions still hold. In these weights, as well as for methods of Poincaré series which could be applied in weights larger that $3/2$ (note that there are no cusp forms of non-positive weight, so modifications have to be made for large weights), note that due to the existence of holomorphic modular forms, the method of matching principal parts won't yield exact formulas. However, the obstruction to obtaining exact formulas will then only have polynomial growth, and if you started with a function with a pole at some cusp, so that there will be rapid growth of the Fourier coefficients, the method of Poincaré series still yields a strong estimate, and, in particular, an asymptotic.