MODULAR FORMS LECTURE 9: MODULAR FORMS: THE BASICS

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> There are five elementary arithmetical operations: addition, subtraction, multiplication, division, and... modular forms.
>
> Martin Eichler

We now return to modular forms. We will see that these will allow us to study ranks of elliptic curves in families of quadratic twists, like E_n.

We have already seen a few examples of modular forms, implicitly. For example, $\wp_\Lambda(z)$ gives modular forms in two ways, where $\Lambda = \langle 1, \tau \rangle$, $\tau \in \mathbb{H}$:

(1) The Taylor coefficients $G_k(\Lambda)$.
(2) **Specialization to torsion points**: $\wp_\Lambda(a + b\tau)$, where $a, b \in \mathbb{Q}$.

Such two-variable functions which encode these two infinite families of modular forms like this are described by the framework of **Jacobi forms**, which can return to later.

Now, we will build up the theory of modular forms. We start with a rigorous definition.

Definition. A **meromorphic modular form** of *weight* $k \in \mathbb{Z}$ is a meromorphic function $f\colon H \to \mathbb{C}$ such that

(1) $f(\gamma\tau) = (c\tau + d)^k f(\tau)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.
(2) f is "meromorphic at ∞". This means the following. Recall that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ implies that $f(\tau + 1) = f(\tau)$ and that this plus meromorphicity implies f has a Fourier expansion

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_n q^n.$$

This is an expansion around ∞; recall that $q \to 0$ as $\tau \to i\infty$ ($e^{2\pi i (i\infty)} = 0$). Meromorphic at ∞ means that $a_n = 0$ for $n \ll \infty$. That is, the Fourier expansion is really

$$f(\tau) = \sum_{n \gg -\infty} a_n q^n = \sum_{n = n_m}^{\infty} a_n q^n$$

for some $n_m \in \mathbb{Z}$. Alternatively, we'll learn to think of these conditions near ∞ as growth rate properties. So we could also require that

$$f(it) = O(e^{Ct})$$

for some $C > 0$ where $t \in \mathbb{R}$ tends to ∞. But we have something even better than that; the beginning of the expansion with non-positive q powers is called the **principal part**, and once you subtract that you have exponential decay.

We now discuss some of the key properties. of modular forms.

Properties/Notation. (1) *The set of all modular forms of weight k is a vector space. We'll denote this by $M_k^{\text{mero}}(\text{SL}_2(\mathbb{Z})) = M_k^{\text{mero}}$.*

(2) *If f is holomorphic on $\mathbb{H}$, then we say f is* **weakly holomorphic**. *The space of such functions is denoted $M_k^!$.*

(3) *If f is holomorphic on $\mathbb{H}$ and at ∞ (meaning $f(\tau) = \sum_{n\geq 0} a_n q^n$), then f is a* **holomorphic modular form**. *We also call holomorphic modular forms simply "modular forms" and we denote the space of them by M_k.*

(4) *If $f \in M_k$ and we further have $a_0 = 0$, that is, $f(\tau) = \sum_{n\geq 1} a_1 q^n$, then we say f is a* **cusp form**. *The space of cusp forms is denoted by S_k (for the German "Spitze").*

(5) *Thus, we have the following containments:*

$$S_k \subseteq M_k \subseteq M_k^! \subseteq M_k^{\text{mero}}.$$

(6) *Modular forms (N.B.: False for more general groups than $\text{SL}_2(\mathbb{Z})$) of odd weight are 0. We can easily see this by considering the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The modularity relation at this matrix implies*

$$f(\tau) = (-1)^k f(\tau) = -f(\tau)$$

for all $\tau \in \mathbb{H}$, and so $f(\tau) \equiv 0$.

(7) *Here is what is special about the factor $(c\tau + d)^k$. If $k = 0$, then the function is invariant under the Möbius transformations, which is natural. Now differentiate*

$$\frac{d}{d\tau}(\gamma \cdot \tau) = \frac{d}{d\tau}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{-2}$$

(we have used that the determinant is 1!). Thus, modularity at γ is the same as

$$\left(\frac{d\gamma\tau}{d\tau}\right)^{\frac{k}{2}} f(\gamma\tau) = f(\tau)$$

which is the same as saying that $f(\tau)(d\tau)^{\frac{k}{2}}$ is invariant. In turn, this is the same as giving a $k/2$ differential form on $\text{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

(8) *Earlier, we claimed that $\text{SL}_2(\mathbb{Z}) \circlearrowleft \mathbb{H}$. Here is a main part of the proof, which is a frequently useful calculation:*

$$\text{Im}(\gamma\tau) = \text{Im}\left(\frac{a\tau + b}{c\tau + d}\right) = \text{Im}\left(\frac{(a\tau + b)(c\overline{\tau} + d)}{|c\tau + d|^2}\right) = \frac{\text{Im}(ad\tau + bc\overline{\tau})}{|c\tau + d|^2} = \frac{(ad - bc)v}{|c\tau + d|^2} = \frac{v}{|c\tau + d|^2} > 0.$$

Two special elements of $\mathrm{SL}_2(\mathbb{Z})$ are

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

These in fact **generate** $\mathrm{SL}_2(\mathbb{Z})$.

Lemma. *We have* $\mathrm{SL}_2(\mathbb{Z}) = \Gamma = \langle S, T\rangle$.

Proof. Let $G = \langle S, T\rangle \leq \Gamma$. The effect of multiplication of powers of S, T is given as follows.

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}, \quad T^n\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix}.$$

Note that $S^2 = -I$ has trivial Möbius transformation action, so we don't really take powers of it. But since $-I$ is in G, we can multiply matrices by a sign. Now let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Suppose $c \neq 0$. If $|a| > |c|$, then do a Euclidean division $a = cq + r$ with $0 \leq r < |c|$. Then

$$T^{-q}\gamma = \begin{pmatrix} a - qc & * \\ c & * \end{pmatrix} = \begin{pmatrix} r & * \\ c & * \end{pmatrix}.$$

Now the upper left entry is $\leq$ the (absolute value) of the lower left entry. Swap these two (with a sign) by multiplying by S. If the lower left entry is non-zero, find another power of T to multiply by to make the lower left value smaller. Eventually, you get a matrix of the form $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$. Since its in Γ, it has to be in the form

$$\begin{pmatrix} \pm 1 & m \\ 0 & \pm 1 \end{pmatrix} = \pm T^m \in G.$$

Thus, $\Gamma \leq G$, and we are done. $\square$

Example 1. This is a constructive proof. A cool way to interpret this is via **continued fractions**. In fact, continued fractions are determined by performing the Euclidean algorithm. For example, say we want to write the matrix $\left(\begin{smallmatrix} 17 & 29 \\ 7 & 12 \end{smallmatrix}\right)$ as $T^*ST^*S\ldots$. How do we find the correct powers? Well, we look at the left column, and we write out the quotient as a continued fraction:

$$\frac{17}{7} = \mathbf{3} - \frac{1}{\mathbf{2} - \frac{1}{\mathbf{4}}}.$$

This corresponds to the representation

$$T^{\mathbf{3}}ST^{\mathbf{2}}ST^{\mathbf{4}}S = \begin{pmatrix} 17 & -5 \\ 7 & -2 \end{pmatrix}.$$

This has the right first column! To find the correct second column, solve

$$\begin{pmatrix} 17 & 29 \\ 7 & 12 \end{pmatrix} = \begin{pmatrix} 17 & -5 \\ 7 & -2 \end{pmatrix} M.$$

This yields

$$M = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = T^2.$$

Thus, we have

$$\begin{pmatrix} 17 & 29 \\ 7 & 12 \end{pmatrix} = T^3 S T^2 S T^4 S T^2.$$