MODULAR FORMS LECTURE 4: MORE ON ELLIPTIC FUNCTIONS

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> Any function addition law is due to an elliptic curve lurking in the background.
>
> Karl Weierstraß

We have seen that $\mathcal{E}_\Lambda$ is a field and is closed under differentiation. We have also constructed a special elliptic function $\wp(z)$. We prove the following result. Note that it gives a practical algorithm for computing this rational function inductively.

Theorem 0.1. *We have that $\mathcal{E}_\Lambda = \mathbb{C}(\wp, \wp')$. That is, all elliptic functions are rational functions in $\wp$ and $\wp'$.*

Proof. Let $f \in \mathcal{E}_\Lambda$. Then we can write $f = g + \wp' h$, where

$$g(z) = \frac{f(z) + f(-z)}{2}$$

and

$$h(z) = \frac{f(z) - f(-z)}{2\wp'(z)}.$$

Note that g and h are even elliptic functions. Thus, it suffices to show that the subfield of all even elliptic functions is $\mathbb{C}(\wp)$.

We do this by **matching zeros and poles**. If $f \in \mathcal{E}_\Lambda$ is even, we list all zeros and poles in Π_Λ.

Note that if z_0 is a zero or pole of f, then so is $-z_0$, as $f(z_0) = f(-z_0)$. Further, if $2z_0 \notin \Lambda$, then the representatives of $z_0, -z_0$ in Π_Λ are distinct. If z_0 is a zero and $2z_0 \in \Lambda$ (that is, z_0 is a 2-*torsion point*), then $f' \in \mathcal{E}_\Lambda$ is an odd function so that

$$f'(z_0) = -f'(-z_0) = -f'(z_0 - 2z_0) = -f'(z_0) \implies f'(z_0) = 0.$$

Thus, a zero at a 2-torsion point is a zero of at least order 2.

Finally, before continuing the proof, we note that for any $a \in \mathbb{C}$, $\wp(z) - \wp(a)$ has a double pole at points in Λ and no other poles. Since the number of zeros (with multiplicity) is equal to the number of poles, there are two zeros of $\wp(z) - \wp(a)$, which are a and $-a$ (distinct if $a \notin \frac{1}{2}\Lambda$, a double zero if $a \in \frac{1}{2}\Lambda$).

Thus, if $z_0 \notin \Lambda$ is a zero of f, the order of vanishing of $\wp(z) - \wp(z_0)$ is **at most** that of f at those zeros. In particular,

$$f_1(z) := \frac{f(z)}{\wp(z) - \wp(z_0)}$$

has two fewer zeros at non-lattice points. The double pole of $\wp(z) - \wp(z_0)$ at 0 gives f_1 order of vanishing two more than f. This introduces no new poles, and no other new zeros (away from Λ).

Since there are only finitely many zeros in Π, repeating this process eventually yields an elliptic function $g(z)$ with zeros only at Λ. Now we get rid of the poles of $g(z)$. We do this by multiplying by factors $\wp(z) - \wp(z_0)$ in the same way. That is, if there is a pole of $g(z)$ at z_0, then we multiply by $\wp(z) - \wp(z_0)$ to lower the order, and continue until all the zeros and poles away from Λ are gone.

Thus, there is a choice of product

$$h(z) = f(z) \cdot \prod_{j=1}^{n} (\wp(z) - \wp(z_0))^{n_j}$$

where $n_j \in \mathbb{Z}$ with no zeros or poles away from Λ. But the number of zeros equals the number of poles in Π_Λ, so there are in fact no zeros or poles of $h(z)$ (i.e., it has trivial divisor). Thus, $h(z)$ is a constant, and so $f(z)$ is a rational function in $\wp$, as claimed.

$\square$

A special differential equation. We now know that all elliptic functions are generated by $\wp, \wp'$. Are there any special relations between the generators $\wp, \wp'$? That is, if I write down distinct rational functions in our theorem above, do I get distinct elliptic functions? The answer connects us with another big topic.

Consider $(\wp')^2$. First, note that $\wp'$ has a triple pole on Λ, and no other poles. It has three simple 0's, specifically at the half-periods $\omega_1, \omega_2, \omega_1 + \omega_2$ (where $\Lambda = \langle \omega_1, \omega_2 \rangle$).

Exercise 1. *Show that $\wp'$ indeed has these three simple zeros.*

Thus, these are all the zeros and poles. The above proof/algorithm implies that $(\wp')^2$ is a cubic polynomial in $\wp$. Indeed, the square $(\wp')^2$ has a double root at the three half-periods and no poles outside of Λ, and so

$$(\wp')^2 = C \cdot \left(\wp(z) - \wp\left(\frac{\omega_1}{2}\right)\right)\left(\wp(z) - \wp\left(\frac{\omega_2}{2}\right)\right)\left(\wp(z) - \wp\left(\frac{\omega_1 + \omega_2}{2}\right)\right)$$

$$= C \cdot \prod_{j=1}^{3} (\wp(z) - \wp(e_i))$$

where C is a constant and e_1, e_2, e_3 are the half-periods. To find C, we only need to check near a single point. We can do this by comparing Taylor expansions at $z = 0$.

First, we know

$$\wp(z) = z^{-2} + O(1)$$

(the residue is zero since the sum of residues is). Thus, we have

$$\wp'(z) = -2z^{-3} + O(1),$$

and the leading term of $(\wp')^2$ is $4z^{-6}$. The leading term of the right hand side of the formula for it in terms of products of $\wp(z) - \wp(e_i)$ is Cz^{-6}. Thus, $C = 4$, and we have shown the following.

Theorem 0.2. *We have*

$$(\wp'(z))^2 = f(\wp(z)), \qquad \textit{where } f(x) := 4\prod_{j=1}^{3}(x - e_i).$$

We can also give another version of this differential equation. We will do this by computing Taylor expansions, which will be very illuminating. Not worrying about the radii of convergence,

Exercise 2. *Fill in the analytic details of convergence in what follows.*

we start with the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + \dots.$$

Differentiating and substituting $x = z/\lambda$ gives

$$\frac{1}{\left(1 - \frac{z}{\lambda}\right)^2} = 1 + \frac{2z}{\lambda} + \frac{3z^2}{\lambda^2} + \dots.$$

Subtracting 1, dividing by λ^2, and using the definition of $\wp(z)$ (after a tiny bit of algebra), we find

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{2z}{\lambda^3} + \frac{3z^2}{\lambda^4} + \frac{4z^3}{\lambda^5} + \dots..$$

As in the (see exercises) proof that the defining sum of $\wp(z)$ is absolutely convergent, this expression is too, and we have

$$\wp(z) = \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + 7G_8 z^6 + \dots,$$

where

$$G_k = G_k(\Lambda) := {\sum_{\lambda \in \Lambda}}' \lambda^{-k}$$

are the **weight** k **Eisenstein series** (the $'$ on the sum will always mean to omit 0). As we'll see soon, these are our first examples of modular forms. Note that the odd powers went away as $G_k = 0$ for k odd upon letting $\lambda \mapsto -\lambda$, though we also knew this would happen since $\wp$ is even. We will shortly see why this is expected in modular forms as well.

Thus, we can build a table

$$\wp(z) = \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + 7G_8 z^6 + \dots,$$

$$\wp'(z) = -\frac{2}{z^3} + 6G_4 z + 20G_6 z^3 + 42G_8 z^5 + \dots,$$

$$(\wp'(z))^2 = \frac{4}{z^6} - 24G_4 \cdot z^{-2} - 80G_6 + (36G_4^2 - 168G_8)z^2 + \dots,$$

$$\wp(z)^2 = \frac{1}{z^4} + 6G_4 + 10G_6 z^2 + \dots,$$

$$\wp(z)^3 = \frac{1}{z^6} + 9G_4 \cdot z^{-2} + 15G_6 + (21G_8 + 27G_4)^2 z^2 + \dots.$$

Taking the notation

$$g_2 := 60G_4, \qquad g_3 := 140G_6,$$

and comparing the above, we find the following.

Theorem 0.3. *We have*

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3.$$

Exercise 3. *Show the details of the proof of this theorem. To do so, do the slightly tedious algebra of comparing coefficients of the difference of LHS and RHS up to the constant term, thus showing it has no zeros or poles. Deduce that the difference of LHS and RHS has no poles and is hence constant. Looking at constant terms, deduce that the constant is zero.*

We will soon see that this result is the key to parameterizing elliptic curves over $\mathbb{C}$.

Combinatorial Application. Another application is the study of convolution divisor sum identities. First, we need one additional fact, which we leave as an exercise similar to the above.

Exercise 4. *Show that* $\wp'' + 30G_4 = 6\wp^2$.

As a corollary of the differential equation and the expansions above, after some basic algebra we find

$$\sum_{n\geq 2}(2n-1)(2n-2)(2n-3)G_{2n}z^{2n-4} + 30G_4$$

$$= 12\sum_{n\geq 2}(2n-1)G_{2n}z^{2n-4} + 6\sum_{p,q\geq 2}(2p-1)(2q-1)G_{2p}G_{2q}z^{2p+2q-4}.$$

Comparing coefficients gives the following recursion.

Theorem 0.4. *We have that*

$$(n-3)(2n+1)(2n-1)G_{2n} = 3 \cdot \sum_{\substack{p,q \geq 2 \\ p+q=n}} (2p-1)(2q-1)G_{2p}G_{2q}.$$

As special cases, we have that $7G_8 = 3G_4^2$ and $11G_{10} = 5G_4G_6$. As a hint of what's to come, if we take $\Lambda = \langle 1, \tau \rangle$, $\tau \in \mathbb{H}$, G_k is a modular form as a function of τ, and its Fourier coefficients are essentially $\sigma_{k-1}(n)$. The relation $7G_8 = 3G_4^2$ is where the following convolution formula we saw comes from (at least once we prove the formula for Fourier expansions of Eisenstein series):

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{0<k<n} \sigma_3(k)\sigma_3(n-k).$$

Thus, this differential equation for $\wp$ implies infinitely many such identities all at once! This recursion will also allow us to show that all modular forms on $\mathrm{SL}_2(\mathbb{Z})$ are generated by just G_4, G_6.

A "surprise" is that there is a sparser recurrence relation of Eisenstein series of a very simple shape only discovered by Romik in 2015 by studying the Witten zeta function for SU(3). These don't come from elliptic functions in the same way, but Mertens and I showed in a joint paper that there is a nice way to view these as coming from modular forms. It would be interesting to search for or prove such things for Eisenstein series on subgroups, or to study what happens for other Witten zeta functions.

As a final "amuse-bouche", you may have wondered what the **zeros** of $\wp$ are. If you are interested, take a glance at the formulas of Eichler-Zagier `https://people.mpim-bonn.mpg.de/zagier/files/doi/10.1007/BF01453974/fulltext.pdf` and of Duke-Imamoğlu `https://www.math.ucla.edu/~wdduke/preprints/zeros.pdf`. It may be surprising that the zeros of $\wp'$, are easily determined (see the exercise above), but for $\wp$ it is definitely not easy to do so explicitly!