LECTURE 29: SPHERE PACKING

LARRY ROLEN, VANDERBILT UNIVERSITY, FALL 2020

She's pulled a Ramanujan.

Thomas Hales, on the sphere packing work of Viazovska, according to this Quanta article: `https://www.quantamagazine.org/sphere-packing-solved-in-higher-dimensions-2016`

1. The problem

The basic question we want to discuss is the following.

Question. *What arrangement of congruent spheres in $\mathbb{R}^d$ is the densest?*

This question is an old one, and has many variations as well as connections with energy minimization problems and coding theory. A nice description of the history of the recent breakthroughs on this problem, due to de Laat and Vallentin, can be found at arxiv: 1607.02111. Some of the history below, and in particular the formulation of some of the statements and proofs below, was also adapted from there.

For $d = 1$, the question is trivial; spheres are line segments and you can fill 100% of the line with them. Let's make some notation for this. Given a **packing** of spheres $\mathcal{P}$, the **finite density** truncated at some radius r is

$$\Delta_{\mathcal{P}}(r) := \frac{\mathrm{Vol}(\mathcal{P} \cap B_d(0,r))}{\mathrm{Vol}(B_d(0,r))},$$

where $B_d(0,r)$ is the ball in $\mathbb{R}^d$ of radius r centered at the origin. The **density** of $\mathcal{P}$ is

$$\Delta_{\mathcal{P}} := \limsup_{r\to\infty} \Delta_{\mathcal{P}}(r).$$

Finally, the **sphere packing constant in dimension** d is

$$\Delta_d := \sup_{\mathcal{P}\subset\mathbb{R}^d} \Delta_{\mathcal{P}}.$$

Since you can fill 100% of $\mathbb{R}^1$ with spheres, we have $\Delta_1 = 1$.

Thue found in 1892 that $\Delta_2 \approx 0.9068$ and that the optimal packing is given by centering the spheres at points in a hexagonal lattice; a more rigorous proof was given by Tóth in 1940. Kepler first asked about the constant Δ_3 when he asked what the best way is to stack cannonballs in a crate. Kepler's problem was solved by Hales in 1998 (it involved many subcases and computer help; he led a full computer verified proof in 2014). Hales showed that $\Delta_3 = \frac{\pi}{\sqrt{18}} \approx 0.7405$. Optimal packings are obtained by stacking layers of hexagons, as one may see in the classic "grocery store orange packing."

Until very recently, these were the only dimensions in which the sphere packing problem was solved. That has now changed thanks to important breakthroughs using Poisson summation and modular forms.

2. Linear Programming Bounds

The first step to the recent progress was the bounds of Cohn and Elkies.

Theorem 2.1 (Cohn-Elkies)**.** *Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a Schwartz function. Further suppose that f and its Fourier transforms satisfy the conditions*

(1) $\widehat{f}(0) = 1$.
(2) $\widehat{f}(u) \geq 0$ *for all* u.
(3) $f(x) \leq 0$ *for all* $|x| \geq R$.

Then the density of any sphere packing in d dimensions is at most

$$\mathrm{Vol}(B_d(0, R/2) \cdot f(0).$$

Remark. *It is very difficult to control inequalities like this for* **both** *a function and its Fourier transform. We will discuss the strategy for how to handle this below.*

Proof. It turns out that the density of any packing can be approximated to arbitrary precision with a periodic packing. So we assume that $\mathcal{P}$ is a periodic packing of spheres of radius $R/2$. Explicitly, this means that

$$\mathcal{P} = \cup_{v \in L} \cup_{i=1}^{m} (v + x_j + B_d(0, R/2))$$

for a lattice L and a set of points $x_1, \ldots, x_m \in \mathbb{R}^d$. It is easy to write down the density of a periodic packing like this, and one finds that it is:

$$\Delta_{\mathcal{P}} = \frac{\mathrm{Vol}(B_d(0, R/2))}{\mathrm{Vol}(\mathbb{R}^d/L)} \cdot m.$$

We now apply Poisson summation. We mainly did this for $\mathbb{Z}$ before, but we mentioned other lattices. In general, it works like it did for $\mathbb{Z}$, but in the summation over the Fourier transform, you sum over the **dual lattice** $L^* := \{y \in \mathbb{R}^d \colon x \cdot y \in \mathbb{Z}\ \forall x \in L\}$ and you have to divide by the volume of the fundamental domain. This yields

$$\sum_{v \in L} \sum_{i,j=1}^{m} f(v + x_j - x_i) = \frac{1}{\mathrm{Vol}(\mathbb{R}^d/L)} \sum_{u \in L^*} \widehat{f}(u) \sum_{i,j=1}^{m} e^{2\pi i u \cdot (x_j - x_i)},$$

where we used a basic translation property of the Fourier transform. Using properties (1) and (2), the right hand side is greater than or equal to $m^2/\,\mathrm{Vol}(\mathbb{R}^d/L)$. We can also apply property (3) to the left hand side. This shows that it is bounded from above by $mf(0)$. Comparing gives the desired result. $\square$

How can we use this to solve the sphere packing problem? We can find lower bounds on Δ_d by finding specific packings with certain densities. If we by some means find a function f which yields an upper bound equal to a packing density we already have, then that packing **must be optimal**. Cohn and Elkies performed computer experiments which suggested that such functions may exist and be findable for $d = 8, 24$. The expected optimal lattices were the famous E_8 and the Leech lattice in dimension 24.

(Coincidentally, the Leech lattice is a very important one which also underpins the Umbral Moonshine of Cheng-Duncan-Harvey, and is closely related to the modular Delta function; recall that theta functions of lattices are modular of weight half the dimension). Cohn and Elkies' experiments found that there were functions which led to upper bounds within 1% of the densities of these lattices, launching the search for these functions.

3. Work of Viazovska

Viazovska created a sensation, when, in a 2017 paper, she found Cohn-Elkies' sought-after function and solved the sphere packing problem in dimension 8. Shortly thereafter, she and collaborators solved the sphere packing problem in dimension 24.

Theorem 3.1 (Viazovska, Cohn-Kumar-Miller-Radchenko-Viazovska)**.** *The lattice* E_8 *and the Leech lattice yield optimal sphere packings in dimensions* 8 *and* 24*.*

How did Viazovska find the functions to plug into the Cohn-Elkies bound? She used modular forms!

The first step is to reduce the problem to searching for a **radial function**, which only depends on the length of the input. Basically, this is because you can average functions f satisfying the conditions of the Cohn-Elkies bound to obtain a function

$$F(x) := \int_{S^{n-1}} f(|x|\xi) d\omega(\xi),$$

where $d\omega$ is the normalized invariant measure on S^{n-1}, and then $F(x)$ also satisfies Cohn-Elkies' conditions and is radial.

The next major idea is to use eigenfunctions of the Fourier transform. Given any radial Schwartz function $F(r)$ (where r is the length of any vector plugged in), we can write it as a linear combination of radial eigenfunctions of the Fourier transform of eigenvalues ± 1. This gives better control over the inequalities we need to satisfy to apply the Cohn-Elkies bound.

There is also an observation that one can make going slightly beyond the proof of the Cohn-Elkies bound we gave above. If a function provides an optimal bound, then it **must** have double zeros at lattice points of length greater than the minimal non-zero length, and a simple zero at the vectors of minimal non-zero length r_0. In the dimension $8, 24$ cases, given the specific lattices that were conjectured to give optimal packings, she needed to pick a function which has double zeros at all points $\sqrt{2k}$ for $k > 1$.

Viazovska brilliantly **forced** this double vanishing at these points by multiplying by the simple factor $\sin^2(\pi r^2/2)$. There are now two different constructions one needs, for what we'll call the "+1 eigenfunction" and the "−1 eigenfunction." In what follows, I'll use some results I wrote down with Ian Wagner in arxiv: 1903.05737; further details of the proofs of the results in the form I'll give here can be found there. This gives a generalization of Viazovska et al's construction. Related work of Feigenbaum, Grabner, and Hardin which is also worth reading alongside this can be found at arxiv: 1907.08558.

4. The +1 eigenfunction case

Viazovska's construction in this case has the following shape (see Proposition 3.2 of R-Wagner):

$$a(r) = -4\sin^2\left(\frac{\pi r^2}{2}\right)\int_0^{i\infty}\phi\left(-\frac{1}{\tau}\right)\tau^{\frac{d}{2}-2}e^{\pi i r^2\tau}d\tau, \qquad (r \geq r_0)$$

Here, $\phi(\tau)$ is a **weakly holomorphic quasimodular form** of weight $k = -d/2+4$, depth 2, and level 1. Quasimodular forms of depth d can also be compactly thought of as polynomials of degree d in E_2 with modular form coefficients. Note that this is essentially a Laplace transform; which is related to the Mellin transform we have studied in the context of L-functions.

Why do quasimodular forms come in, and specifically why of depth 2? This is because (see Proposition 3.1) the function can then be extended to all $r \geq 0$ by the more complicated function

$$a(r) = \int_{-1}^{i}\phi\left(-\frac{1}{\tau+1}\right)(\tau+1)^{\frac{d}{2}-2}e^{\pi i|x|^2\tau}d\tau + \int_{1}^{i}\phi\left(-\frac{1}{\tau-1}\right)(\tau-1)^{\frac{d}{2}-2}e^{\pi i|x|^2\tau}d\tau$$
$$-2\int_0^i\phi\left(-\frac{1}{\tau}\right)\tau^{\frac{d}{2}-2}e^{\pi i|x|^2\tau}d\tau + 2\int_i^{i\infty}\phi(z)e^{\pi i|x|^2\tau}d\tau.$$

Now as long as $\phi(\tau)$ is 1-periodic and satisfies a good growth condition as $\tau \to i\infty$, this last expression is a Schwartz function which is an eigenfunction of the Fourier transform of eigenvalue $-i^{-\frac{d}{2}}$ (this is a direct calculation). Modular forms, and quasimodular forms, are exactly the types of things that have nice asymptotics near $i\infty$ and are translation invariant. When the $\sin^2$ factor is written in terms of an exponential and that integral is expanded out in a particular way, one finds that a quasimodular form of depth 2 is exactly what's needed. Moreover, for large enough r, it had double zeros at the right places by choice. It would have a double zero at $r = r_0$, but its supposed to have a simple zero there; however, we required the form to be weakly holomorphic, that is, to have a pole which cancels out one of these zeros.

5. The −1 eigenfunction case

This works in a somewhat similar, but also a bit different, way. Now, one needs forms on $\Gamma(2)$, instead of $\Gamma(1)$. Recalling that slashing with matrices gives new modular forms, even when you're outside of the group of modular transformations, Viazovska considered forms which satisfy the particular relationship

$$\Psi = \Psi|S + \Psi|T.$$

Then a little bit different of a construction yields the desired eigenfunction by a similar construction (Viazovska used theta functions in this case).

6. The specific choices of functions ϕ and ψ

With this setup, the question is: can one find forms ϕ and ψ to make the Viazovska constructions work, and to yield the correct Cohn-Elkies bound? Of course the answer is yes. Here is one way you could find them.

In the first case, to find ϕ, similar to what we've done before, our weakly holomorphic quasimodular form must be a (holomorhpic) quasimodular form divided by a power of $\Delta(\tau)$. Basically, one then wants to find a form of depth 2, level 1, weight $-\frac{d}{2}+4+12n$ which is $O(q^{n+1})$ for as small an n as possible. This is a finite dimensional space; all quasimodular forms are polynomials in E_2, E_4, E_6, so one can just perform a finite search. For instance, with $d=8$, the function which gives Viazovska's example is

$$E_2^2E_4^2 - 2E_2E_4E_6 + E_6^2.$$

For the other eigenfunction, one can also multiply by a power of delta to land in a space of holomorphic forms, and everything is finite dimensional, so in any particular case one can simply search for the best possible functions satisfying the conditions. For example, with $d=8$, this holomorphic form which is optimal and gives Viazovska's example is

$$\frac{1}{3}(U^2-V^2)E_6 + \frac{2}{3}WE_4^2,$$

where $U=\theta_3^4$, $V=\theta_2^4$, $W=\theta_4^4$ are the fourth powers of the classical Jacobi theta functions:

$$\theta_2(\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i(n+1/2)^2\tau},$$
$$\theta_3(\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau},$$
$$\theta_4(\tau) := \sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i n^2\tau}.$$

7. Final Thoughts

It has been a fun semester with all of you. It has been a great experience, and I'm glad that many people from different math backgrounds joined in. Hope you have a great break and that I'll see you all around!