# MODULAR FORMS LECTURE 3: EXAMPLES OF ELLIPTIC FUNCTIONS

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> I turn with terror and horror from this lamentable scourge of functions with no derivatives.
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> Charles Hermite, on Weierstraß' examples of continuous, nowhere-differentiable functions. See also Riemann's example, which is basically an antiderivative of a modular form: $\sum_{n\geq 1} \frac{\sin(2\pi n^2 x)}{n^2}$.

Last time, we proved some basic structure theorems on the space of elliptic functions. We then asked whether any "interesting" examples exist. The answer is yes!

To build such functions, we follow a general procedure. This idea is very common, and we'll see it later in modular forms. Anytime I have a function which has some family of symmetry relations, its good to try to represent this as a function invariant under some group action. In this case, we have a group action of the lattice acting on $\mathbb{C}$ by translation. That is, $\Lambda \circlearrowleft \mathbb{C}$, where $\lambda \cdot z := z + \lambda$. Further, $\Lambda$ acts on the set of functions $f\colon \mathbb{C} \to \mathbb{C}$, by $(\lambda \cdot f)(z) := f(z+\lambda)$. An doubly periodic function is precisely a function which is invariant under this action.

A very common idea in algebra, used in many topics is the following. If you have a group action and want an invariant element, you can **average** over the whole group. That is, you can take a function and add up all its images under the action. For instance, if we take a **kernel function** some meromorphic $\varphi\colon \mathbb{C} \to \mathbb{C}$, then you can define the sum of translates function:

$$f_\varphi(z) := \sum_{\lambda\in\Lambda} \varphi(z+\lambda).$$

The key issue to watch out for is absolute convergence. Assuming that we can rearrange terms in the sum, then we get double periodicity for $f_\varphi$ **for free:**

$$f_\varphi(z+\lambda') = \sum_{\lambda\in\Lambda} \varphi(z+\lambda'+\lambda) = \sum_{\lambda''\in\Lambda} \varphi(z+\lambda''),$$

since as $\lambda$ runs over $\Lambda$, the set of elements $\lambda' + \lambda$ does too, just in another order.

We analyze this where we take as choice of kernel function $\phi(z) = z^{-k}$ for some $k \in \mathbb{N}$. This gives us the functions

$$f_k(z) := \sum_{\lambda\in\Lambda} \frac{1}{(z+\lambda)^k},$$

which are elliptic functions **assuming** absolute convergence. This turns out to work for $k \geq 3$.

This matches with what we saw last time: non-constant elliptic functions have to have at least 2 poles. Thus it is to be expected that $f_1$ doesn't converge. For $f_2$, absolute convergence is "almost" satisfied, and indeed there is a modified version with only 2 poles which "fixes" the issue (and is basically an antiderivative of $f_3$).

Its so special that there is a Wikipedia page not only devoted to this function, but to its special typographical symbol. It is also the function pictured on the course website. This is the **Weierstraß $\wp$-function**:

$$\wp(z) = \wp_\Lambda(z) := \frac{1}{z^2} + \sum_{\lambda\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right).$$

**Exercise 1.** *Prove that this converges absolutely and uniformly on compact sets (away from the poles in $\Lambda$). As a hint, consider the term*

$$\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} = \frac{2z - z^2/\lambda}{(z-\lambda)^2\lambda}.$$

*For $|\lambda|$ large, this behaves like $\frac{1}{|\lambda^3|}$, and it is summed up in two directions.*

**Exercise 2.** *Now show that $f_k(z)$ is an elliptic function for $k \geq 3$.*

**Theorem.** *We have that $\wp_\Lambda(z) \in \mathcal{E}_\Lambda$.*

*Proof.* Firstly, we've seen that $f_3(z)$ is elliptic. Now we observe that $\wp'(z) \doteq f_3(z)$. Now suppose that $\Lambda$ is generated as $\Lambda = \langle\omega_1, \omega_2\rangle$. Then for $i = 1, 2$, we have

$$\frac{d}{dz}\left(\wp(z+\omega_i) - \wp(z)\right) = \wp'(z+\omega_i) - \wp'(z) = f_3(z+\omega_i) - f_3(z) = 0.$$

Thus,

$$\wp(z+\omega_i) - \wp(z) = C_i \in \mathbb{C}$$

is a constant. But, $\wp(z)$ is an **even function** (letting $z \mapsto -z$ and $\lambda \mapsto -\lambda$ preserves the sum above). Taking the value $z = -\omega_i/2$, we have

$$C_i = \wp(\omega_i/2) - \wp(-\omega_i/2) = 0.$$

Thus,

$$\wp(z+\omega_i) - \wp(z) = 0.$$

□

By inspection, $\wp$ also has a double pole at lattice points and nowhere else. Thus, $\wp$ is the smallest (in the sense of poles) elliptic function we could have, and has a pole at only one location in the fundamental domain. This suggests that you could try to build up elliptic functions using $\wp$. A clear obstruction to building up all elliptic functions is that $\wp$ is even. But then $\wp'$ is odd, and it turns out that all elliptic functions can be built out of $\wp, \wp'$. We record this amazing result for reference, and will prove it next time.

**Theorem.** *We have that*

$$\mathcal{E}_\Lambda = \mathbb{C}(\wp, \wp').$$

*That is, all elliptic functions are rational functions in $\wp, \wp'$.*

Before proving this result, we give a bit of history on elliptic functions. As expected, the word "elliptic" comes from the word "ellipse". In the history/study of complex analysis, after the elementary functions (rational functions, trig functions, etc.), one of the "next" functions to study is the $\Gamma$-function. In the late $18th$ and early 19-th centuries, people began studying integrals of the form

$$F(x) = \int_a^x \frac{dt}{\sqrt{p(t)}},$$

where $p(t)$ is a real degree 3 or 4 polynomial. These came up in computations of arc lengths of ellipses, as well as other curves like lemniscates.

**Elementary Lemma.** *The study of these integrals can be reduces to the case when $p(t) = 4t^3 - \alpha t - \beta$ (Weierstraßnormal form).*

A strange formula due to Fagnano arises when $p(t) = 1 - t^4$.

**Theorem** (Fagnano)**.**

$$2F(x) = F\left(2x \cdot \frac{\sqrt{1-x^4}}{1+x^4}\right).$$

**Corollary.** *The lemniscate ($\infty$) can be doubled with straight edge and compass!*

Euler proved the following extension.

**Theorem** (Euler's Addition Formula for Lemniscates)**.** *We have*

$$F(x) + F(y) = F\left(\frac{x\sqrt{1-y^4} + y\sqrt{1-x^4}}{1+x^2y^2}\right).$$

These are strange transformation formulas.

**Idea** (Abel, greatly furthered by Jacobi)**.** Consider instead the **inverse functions**.

When one does this, things look much nicer, and this gives rise to the elliptic functions.

The history of elliptic functions is like if mathematicians hadn't discovered $\sin, \cos$, but studied arc lengths of circles first. As we know, $\sin, \cos$ are "nicer", and have nicer looking properties/formulas than $\arcsin, \arccos$. For many more details on this history, an excellent reference is Köcher-Krieg's "Elliptische Funktionen und Modulformen". This wasn't ever translated to English, but if you can read some mathematical German, its a great read which goes into great detail about this history and covers more on the elliptic functions than most standard books on modular forms.

**Question.** *Where will modular forms come in?*

Earlier, we required that $\omega_1/\omega_2 \notin \mathbb{R}$. We can do more. WLOG, we can usually rescale so that $\Lambda = \langle 1, \tau \rangle$, and require that $\mathrm{Im}(\tau) > 0$, i.e., that $\tau \in \mathbb{H}$. The Taylor coefficients of $\wp_{\langle 1,\tau \rangle}$, now functions of $\tau$, will turn out to be modular forms. Thus, as we saw before, modular forms are often generating functions for interesting sequences, and $\wp$ is a generating function for interesting *modular forms*.