LECTURE 29: MOONSHINE

LARRY ROLEN, VANDERBILT UNIVERSITY, FALL 2020

> I was over the moon when I proved the moonshine conjecture", and "I sometimes wonder if this is the feeling you get when you take certain drugs. I don't actually know, as I have not tested this theory of mine.
>
> ---
>
> Richard Borcherds, according to Roberts, Siobhan (2009), King of Infinite Space: Donald Coxeter, the Man Who Saved Geometry, Bloomsbury Publishing USA, p. 361

Moonshine is a remarkable connection between modular forms and algebra. These days, it has grown into a larger topic. However, for most of today, we'll focus on the special case of **monstrous moonshine**. The word "moonshine" besides the usual definition of illegally manufactured liquor, also denotes something which is foolish or crazy. Conway and Norton chose this term because at the outset, the connections seemed so strange that they were almost "crazy" to believe. The term "monster" refers to the **monster group**, which we'll now discuss.

1. Classification of finite simple groups

One of the crowning achievements of modern algebra is the **classification of finite simple groups**. Simple groups (recall that these are those with no non-trivial proper normal subgroups) are the "building blocks" of finite group theory, similar to how the prime numbers are the "atoms" of the integers. More precisely, the famous **Jordan-Hölder Theorem** states that any finite group is built out of a unique sequence of simple groups. Thus, in order to understand finite group theory in general, a first major goal is to find all finite simple groups.

Theorem 1.1 (Classification of Finite Simple Groups)**.** *Let G be a finite simple group. Then G is isomorphic to one of the following groups:*

1. *The cyclic group $\mathbb{Z}/p\mathbb{Z}$ for a prime number p.*
2. *The alternating group A_n for $n \geq 5$.*
3. *A simple group of "Lie type."*
4. *The Tits group of order 17971200.*
5. *One of 26 "sporadic groups" that don't fit into any of the above categories.*

This proof was extremely difficult, and required input by many many mathematicians. In fact, it spanned tens of thousands of pages in hundreds of papers of about 100 authors. A proof was claimed in 1983, but this had a gap. That gap was filled by a 1221 page paper in 2004.

The largest of the sporadic groups is the **Monster Group** $\mathbb{M}$. It is indeed quite large, with order

$$|\mathbb{M}| = 2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 8 \cdot 10^{53}.$$

It also contains most of the sporadic examples inside of it, as 20 of the 26 groups are quotients of subgroups of the monster.

2. Strange properties of the monster

Even before this group was proven to exist, it was conjectured to satisfy some strange "coincidences." This is where modular forms come into the picture.

The connection with modular forms can already be observed by looking at the order of the group. The primes that show up in the factorization of this number, listed above, are very important. This is because

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}$$

is precisely the set of primes p for which

$$\Gamma_0(p)^+,$$

the extension of $\Gamma_0(p)$ by the Fricke involution W_p, has genus 0. That is, for which the corresponding **modular curves** $\Gamma_0(p)^+\backslash\widehat{\mathbb{H}}$ (recall that $\widehat{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$) are genus 0 compact Riemann surfaces. Fittingly, Ogg offered a bottle of Jack Daniels whiskey for an explanation of this "coincidence."

This already suggests a connection to modular functions. A general fact is that genus zero implies that the set of functions on the curve, in this case, modular functions for $\Gamma_0(p)^+$, are generated by a single element. We saw this before in the case of $\mathrm{SL}_2(\mathbb{Z})$, where we proved that all level 1 modular functions are rational functions in the j-invariant. If we normalize such a generator so that its constant term is zero and it has leading coefficient 1, then its unique, and we call this function the **Hauptmodul**, or principal modulus for $\Gamma_0(p)^+$. For instance, the Hauptmodul in level 1 is the basic modification of the j-invariant which we denote by J:

$$J(\tau) := j(\tau) - 744 = q^{-1} + 196884q + O(q^2).$$

This number 196884 is a big hint in disguise. McKay started down the path that eventually led to moonshine by making the observation that

$$196884 = 196883 + 1.$$

The left hand side is the q^1 coefficient of the Hauptmodul for $\mathrm{SL}_2(\mathbb{Z})$, as we saw. The significance of 196883 is that it is the smallest dimension of a non-trivial irreducible representation (irrep) of the monster $\mathbb{M}$. Of course, 1 is also a dimension of an irrep, the trivial one! Thus, the first interesting coefficient of $J(\tau)$ is a sum of dimensions of irreps of $\mathbb{M}$.

Surely this is just a coincidence! After all, what could the relation possibly be? On the other hand, these numbers are pretty big, and extremely close. And the set of primes above was also a fairly specific list of primes. However, if we look at further coefficients of $J(\tau)$, similar patterns continue. For example, the next coefficient, of q^2, is

$$21493760 = 21296876 + 196883 + 1,$$

and 21296876 is also a dimension of an irrep of $\mathbb{M}$. The same holds if we look at the coefficient of q^3:

$$864299970 = 842609326 + 21296876 + 2 \cdot 196883 + 2 \cdot 1,$$

when we note that 842609326 is also a dimension of an irrep of the monster, and the smaller dimensions are now added with some multiplicities. But its still a pretty simple relationship, and something must really be going on now!

3. Thompson's idea

McKay became convinced that these formulas carried some meaning, and wrote to Thompson. Thomspon suggested that such equations must indicate that there is an infinite dimensional graded $\mathbb{M}$-representation

$$V = V_{-1} \oplus V_1 \oplus V_2 \oplus V_3 \oplus \cdots$$

with $V_{-1} = \rho_0$, $V_1 = \rho_1 \oplus \rho_0$, $V_2 = \rho_2 \oplus \rho_1 \oplus \rho_0$, $V_3 = \rho_3 \oplus \rho_2 \oplus \rho_1 \oplus \rho_1 \oplus \rho_0 \oplus \rho_0$ decompositions in terms of the irreps $\rho_0, \rho_1, \ldots$ of $\mathbb{M}$. This could provide an explanation of McKay's observation if the coefficients of $J(\tau)$ are dimensions of the components:

$$J(\tau) = \dim(V_{-1}) + \sum_{n \geq 1} \dim(V_n) q^n.$$

Thompson went further. Given any element $g \in \mathbb{M}$, we can consider the character of the representation ρ given by

$$\mathrm{ch}_\rho(g) := \mathrm{tr}(\rho(g)),$$

that is, by applying the representation and then taking a trace. There are 194 conjugacy classes of $\mathbb{M}$, and so working with these characters, while a transformed set of information, is much easier than working with, in the smallest non-trivial case, 196883×196883 matrices! This leads to the **McKay-Thompson series**

$$T_g(\tau) := \mathrm{ch}_{V_{-1}}(g) q^{-1} + \sum_{n \geq 1} \mathrm{ch}_{V_n}(g) q^n.$$

As a special case, if g is the identity, then $\rho(g)$ is the identity matrix, and so the trace is simply the dimension of the representation. Thus, T_{Id} recovers the series above that we believe should be $J(\tau)$.

4. Full moonshine

Conway and Norton followed up on these ideas and discovered that many of the McKay-Thompson series T_g are in fact Hauptmoduln. This led to the following conjecture which they termed **Monstrous Moonshine**.

Conjecture 1. *For any element $g \in \mathbb{M}$, the McKay-Thompson series $T_g(\tau)$ is a Hauptmodul for some (specified) group $\Gamma_g \leq \mathrm{SL}_2(\mathbb{R})$.*

The next major step was that Atkin-Fong-Smith proved that Thompson's graded representation V exists and that the McKay-Thompson series are the correct Hauptmoduln; however, this allowed for the possibility that the representation is **virtual** (a virtual representation is an integer linear combination of representations).

Later, Frenkel-Lepowsky-Meurman proved Thompson correct by showing that the representation V really does exist.

Finally, Borcherds tied up the classical story with the following huge result (he won the Fields Medal for this and other work).

Theorem 4.1 (Borcherds)**.** *The Monstrous Moonshine Conjecture of Conway and Norton is true.*

This was a sensational proof, and introduced revolutionary new methods. Borcherds wrote down **vertex algebras**, complicated objects inspired by physics, which Frenkel, Lepowsky, and Meurman modified to obtain **vertex operator algebras** (VOAs). The definitions (and even giving a single non-trivial example) are very long, so we'll have to skip them here. However, there is a close relationship between VOAs and conformal field theories from string theory, and the relationship of these objects to modular forms has opened up a lot of doors. There are now many collaborations between modular forms people and physicists which these connections made possible.

5. New theories of moonshine

All of the above is for the monster. However, there are other sporadic simple groups. What about those? These turn out to be connected to modular objects as well! There are several extensions, for example the **Umbral Moonshine Conjecture** of Cheng, Duncan, and Harvey from 2012, which was a cohesive set of 23 conjectured infinite-dimensional representations. The first case was proven by Gannon, and the remaining ones by Duncan, Griffin, and Ono. Here, Hauptmoduln are replaced by more complicated objects known as **mock modular forms**. However, a "VOA-type" algebraic construction is still open, and if found, is likely to have wide-ranging applications. There are also various arithmetic applications one can consider, for example to central L-values of elliptic curves.

Thus, moonshine is a very surprising and amazing story, which has led to large sets of new mathematics and physics, and will certainly lead to many more important theories in the years to come.