LECTURE 27: HYPERBOLIC EXPANSIONS, THE SHIMURA AND SHINTANI LIFTS, AND TUNNELL'S THEOREM ON THE CONGRUENT NUMBER PROBLEM

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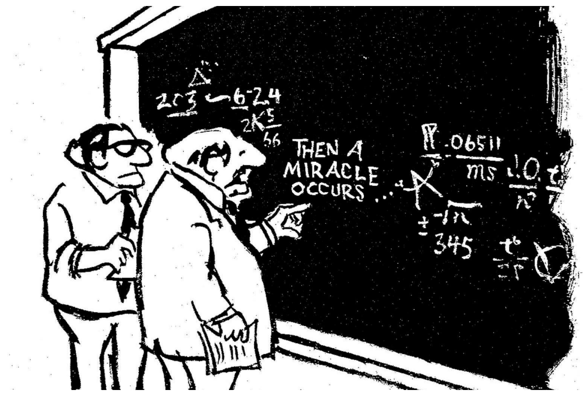

Sidney Harris, 1977

1. Poincaré series

To study central L-values of even weight newforms in families of quadratic twists, we need the **Shimura/Shintani-correspondence**. This is a special map between integer and half-integer weight modular forms. We will sketch one method for constructing this correspondence which was given by Kohnen.

This construction will rely on two types of **Poincaré series**. Recall that a Poincaré series is a modular form constructed as an average of slashing operators over the modular group. Our prime example of this so far has been the Eisenstein series

$$E_k(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} 1|_k \gamma.$$

The generalized version of this, say to congruence subgroups $\Gamma_0(N)$, are the Poincaré series for the **seed** φ:

$$P_{k,N}(\varphi; \tau) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \varphi|_k \gamma.$$

As we saw, this is automatically modular of weight k if φ has the right asymptotics to make this converge absolutely. It turns out that requiring

$$\varphi(\tau) = O(v^{k-2+\varepsilon})$$

is sufficient. For instance, in $k = 2$, the seed function $\varphi = 1$ which gives the Eisenstein series E_2 almost works, as we saw, and for $k \geq 4$ even, the function $\varphi = 1$ giving the Eisenstein series E_k works.

The Eisenstein series is the first case in an important family of Poincaré series.

Definition. The **holomorphic Poincaré series of exponential type** are the series with seed function $\varphi(\tau) = q^m$, with $m \in \mathbb{Z}_{\geq 0}$:

$$P_{k,N,m}(\tau) := P_{k,N}(q^m; \tau) \in M_k(N)$$

If $m = 0$, then this is an Eisenstein series, and if $m > 1$, this is a cusp form.

These series for all m span $M_k(N)$. Since there are infinitely many Poincaré series and this is a finite-dimensional space, there are infinitely many relations among the Poincaré series; however, these relations aren't well-understood.

2. Three types of expansions of modular forms

Poincaré series are implicitly related to q-expansions (expansions around $i\infty$), or expansions at other cusps (you need these for example to construct Eisenstein series which span the part of M_k orthogonal to cusp forms in general level N). These are called **parabolic expansions**. Correspondingly, the Poincaré series we've just given are sometimes called parabolic Poincaré series. Since the cusps are distinguished points, namely, they are compactification points for $\Gamma\backslash\mathbb{H}$ and there are only finitely many of them, expansions around these points are quite natural. However, it turns out that there are other sorts of expansions too, which were developed into a nice theory in general by Petersson.

There are also **elliptic expansions**, which are essentially Taylor series of modular forms around points in $\mathbb{H}$. We will soon be studying the leading term in these expansions, namely the **values** of modular forms at points in the upper half plane. Specifically, we'll see that there is much deep structure of such values at **CM points**; imaginary quadratic numbers in $\mathbb{H}$.

The final type of expansion, which will be relevant for us here, are the **hyperbolic expansions** (you may have guessed the name from the terms parabolic and elliptic above). The language for these expansions comes from the following characterization. A matrix in $\gamma \in \mathrm{SL}_2(\mathbb{R})$ is elliptic if $|\mathrm{tr}(\gamma)| < 2$, parabolic if $|\mathrm{tr}(\gamma)| = 2$, and hyperbolic if $|\mathrm{tr}(\gamma)| > 2$.

We can see what's special about this categorization through the lens of fixed points. Given a matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, a point $\tau \in \mathbb{H} \cup \mathbb{R} \cup \{i\infty\}$ is fixed if

$$\frac{a\tau + b}{c\tau + d} = \tau \implies a\tau + b = c\tau^2 + d\tau \implies c\tau^2 + (d-a)\tau - b = 0.$$

The discriminant of this quadratic equation is

$$(d-a)^2 + 4bc = \mathrm{tr}(\gamma)^2 - 4\det(\gamma) = \mathrm{tr}(\gamma)^2 - 4.$$

Thus, if γ is parabolic, the discriminant is 0, and so there is just one fixed point, at $\frac{a-d}{2c}$. This expansion is related to Fourier expansions at the cusps, which are fixed points of parabolic matrices. If γ is elliptic, then the discriminant is *negative*, so there is one fixed point of γ in the upper half plane, and one in the lower half plane. This is used to study an expansion around this point in $\mathbb{H}$; later to study values of modular forms at CM points, we'll make critical use of the matrix fixing these points. Finally, if γ is hyperbolic, then the discriminant is positive, and so the fixed points are **a pair of real quadratic points** on $\mathbb{R}$. Hyperbolic expansions are thus related to expansions around pairs of real quadratic points.

Remark. *The terminology is meant to be reminiscent of the classification of plane conics* $ax^2+bxy+cy^2 = 1$ *in terms of their discriminants* b^2-4ac*; if the discriminant is negative, its an ellipse, if the discriminant is* 0*, its a parabola, and if the discriminant is positive, its a hyperbola.*

An excellent reference with many nice pictures and explanations about the trifecta of expansions of modular forms, by O'Sullivan and Imamoglu, can be found here: `https://arxiv.org/pdf/0806.4398.pdf`

3. Connection with quadratic forms and hyperbolic Poinaré series

Looking in more detail at the hyperbolic situation, given a pair of real quadratic points $\eta = (\eta_1, \eta_2)$ fixed by a hyperbolic matrix $\gamma_\eta = \left(\begin{smallmatrix} a & b \\ c & b \end{smallmatrix}\right)$, the equation we gave above when solving for fixed points gives an associated quadratic form:

$$Q_{\gamma_\eta}(X, Y) = cX^2 + (d - a)XY - bY^2$$

of discriminant $D = |a + d|^2 - 4 > 0$ with $Q(\tau, 1) = 0$ for $\tau = \eta_1, \eta_2$. We also denote by Γ_η the group of all hyperbolic matrices fixing η.

Following Katok we define on a subgroup Γ a set of **hyperbolic Poincaré series** by (the notation "ϑ" does not mean that this is a theta function, but is just the notation for this in the literature) :

$$\vartheta_{\Gamma,k,\gamma_\eta}(\tau) := \sum_{\gamma \in \Gamma_\eta \backslash \Gamma} \left(Q_{\gamma_\eta}(\tau)^{-\frac{k}{2}} \right) \big|_k \gamma.$$

Katok showed that $\vartheta_{\Gamma,k,\gamma_\eta} \in S_k(\Gamma)$, and, moreover, that like the parabolic Poincaré series above, these also span the space of cusp forms.

A related function, which can be built out of Katok's Poincaré series, is a very famous function of Zagier, defined for $D > 0$ congruent to 0 or 1 mod 4 and integral $k \geq 2$:

$$F_{k,D}(\tau) := \sum_{\substack{b^2-4ac=D \\ (a,b,c)=1}} \frac{1}{(a\tau^2 + b\tau + c)^k}.$$

Exercise 1. *Show that Zagier's* $F_{k,D}$ *function is modular by a direct calculation! That is, show directly that* $F_{k,D} \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$.

4. Relationship between the expansions

Given these three expansions of modular forms, it is natural to ask how they relate. For instance, one can take Poincaré series with respect to one expansion, and then ask about the expansions of the resulting modular forms with respect to any of the above theories. There turn out to be very deep relationships here. We will look at one uncovered by Kohnen, which gives an explicit way to view the Shimura/Shintani correspondence.

For this, we need some modifications. Firstly, we need the **genus character**. This has a complicated definition, but is important in many applications. In fact, the "strange" looking function χ that was mentioned in one criterion for congruent numbers in the epigraph at the top of the last set of notes was a specific evaluation of a genus character in disguise. Given a discriminant $D \equiv 0, 1 \pmod 4$, and given a quadratic form $Q = Q(x,y) = [a,b,c] = ax^2 + bxy + cy^2$ in $\mathcal{Q}_D$, the set of binary, integral ($a, b, c \in \mathbb{Z}$) quadratic forms of discriminant D, the genus character is given by

$$\omega_D(Q) := \begin{cases} 0 & \text{if } (a,b,c,D) > 1 \\ \left(\frac{D}{r}\right) & \text{if } (a,b,c,D) = 1 \text{ and } Q \text{ represents } r \text{ with } (r,D) = 1. \end{cases}$$

Here, by "Q represents m," we mean that there are $x, y \in \mathbb{Z}$ with $Q(x,y) = m$.

Exercise 2. *Show that this is in fact-well-defined by showing that if Q represents both r and s, then $4rs$ can be written as $x^2 - Dzy^2$ for $x, y \in \mathbb{Z}$, and concluding (you may want to refer to basic facts about Kronecker symbols if you haven't worked with them much before) that*

$$\left(\frac{D}{r}\right) = \left(\frac{D}{s}\right).$$

Now we have an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{Q}_D$ given by

$$Q \circ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (x,y) := Q(\alpha x + \beta y, \gamma x + \delta y).$$

Exercise 3. *Show that $\omega_D(Q)$ is a function on* $\mathrm{SL}_2(\mathbb{Z})$**-classes**.

Then we can modify Zagier's original $F_{k,D}$ function to a version with level and twisted by a genus character. Specifically, for $k \geq 2$, $N \geq 1$, $D, D' \equiv 0, 1 \pmod 4$, $z \in \mathbb{H}$, and $DD' > 0$, we set

$$f_{k,N}(z; D, D') := \sum_{\substack{b^2 - 4ac = DD' \\ N \mid a}} \omega_D([a,b,c])(az^2 + bz + c)^{-k} \in S_{2k}(N).$$

In order to talk about the connection with elliptic curves, and hence access the Congruent Number Problem, we need to consider the case of weight 2 cusp forms, that is, when $k = 1$. The definition above is not absolutely convergent for $k = 1$, but it is for any $k > 1$. Thus, we can use a **Hecke trick** to extend to weight 2, as Kohnen did. Consider

$$f_{1,N}(z; s, D, D') := v^s \sum_{\substack{b^2-4ac=DD' \\ N|a}} \omega_D([a,b,c])(az^2+bz+c)^{-1}|az^2+bz+c|^{-s}, \quad (\mathrm{Re}(s) > 0).$$

Then it turns out this has an analytic continuation to $s = 0$, which we denote $f_{1,N}(z; D, D')$, which is cuspidal if N is **cube-free**. The fact that this depends on cubes of primes dividing the level is a hint that this calculation has some real work in it!

The magic happens when we take **generating functions**.

Definition. For N odd, $k \geq 1$, D a **fundamental discriminant** (this means that $D = 1$ or is a discriminant of a quadratic field; or more elementarily, $D \equiv 1 \pmod 4$ and is square-free or $D = 4m$ with $m \equiv 2, 3 \pmod 4$ with m square-free), and the sign of D chosen so that $(-1)^k D > 0$, and with $q = e(\tau)$, we define

$$\Omega_{k,N}(z, \tau; D) := i_N c_{k,D}^{-1} \sum_{\substack{m \geq 1 \\ (-1)^k m \equiv 0,1 \pmod 4}} m^{k-\frac{1}{2}} \left(\sum_{t|N} \mu(t) \left(\frac{D}{t}\right) t^{k-1} f_{k,\frac{N}{t}}(tz; D, (-1)^k m) \right) q^m.$$

Here i_N is the index of $\Gamma_0(N)$

$$i_N := [\mathrm{SL}_2(\mathbb{Z})\colon \Gamma_0(N)]$$

and

$$c_{k,D} := (-1)^{\left\lfloor \frac{k}{2} \right\rfloor} |D|^{-k+\frac{1}{2}} \pi \binom{2k-2}{k-2} 2^{-3k+2}.$$

As a function of z, this is in $M_{2k}(N)$, as the z-dependence is only in the $f_{k,N}$ functions. Moreover, by the above, its cuspidal if $k \geq 2$ or N is cube-free. Amazingly, it is a modular form in the τ variable as well, but **of a different weight**.

Theorem (Kohnen)**.** *Letting $\zeta := e(z)$, for $z \in \mathbb{H}$, we have*

$$\Omega_{k,N}(z, \tau; D) = i_N c_{k,D}^{-1} \frac{(-1)^{\left\lfloor \frac{k}{2} \right\rfloor} (2\pi)^k \cdot 3}{(k-1)!} \sum_{n \geq 1} n^{k-1} \left(\sum_{\substack{d|n \\ (d,N)=1}} \left(\frac{D}{d}\right) (n/d)^k P_{k,N,\frac{n^2|D|}{d^2}}(\tau) \right) \zeta^n.$$

This is thus a cusp form of half-integral weight in $S_{k+\frac{1}{2}}(4N)$.

The proof of this theorem is done by computing Fourier expansions (for example, for the parabolic Poincaré series, one has to generalize our previous calculation of Fourier expansions of Eisenstein series, and one must do the same for the hyperbolic Poincaré series), and then by formally manipulating the sums in a brilliant manner. A miracle then occurs, allowing one to express this two variable gadget as a weight $2k$ modular form in one variable and a weight $k+1/2$ modular form in the other. Thus, the correctly "decorated" generating function of hyperbolic Poincaré series of one weight is a generating function of parabolic Poincaré series of another weight.

We can use this function to shuffle between weights $2k$ and $k+1/2$.

Definition. Given $g \in S_{k+\frac{1}{2}}(4N)$, the **Shimura lifts** are given by integrating using the Petersson inner product:

$$g|\mathcal{S}_{k,N,D}(z) := \langle g, \Omega_{k,N}(-\overline{z}, \bullet; D)\rangle \in S_{2k}(N).$$

Here $\bullet$ denotes the variable integrated over. The **Shintani lifts** of an even weight form $f \in S_{2k}(N)$ are given by integrating against the other variable of $\Omega_{k,N}$:

$$f|\mathcal{S}^*_{k,N,D}(\tau) := \langle f, \Omega_{k,N}(\bullet, -\overline{\tau}; D)\rangle \in S_{k+\frac{1}{2}}(4N).$$

This was not the original definition of Shimura and Shintani, but it is an equivalent one, and essentially a "holomorphic projection" of previous kernel functions for the Shimura/Shintani lift. A more classical and direct way to write the Shimura lift that's worth mentioning is the following, which is explicit but whose modularity properties aren't clear:

$$g|\mathcal{S}_{k,N,D} = \sum_{n\geq 1} \sum_{\substack{d|n \\ (d,N)=1}} \left(\frac{D}{d}\right) d^{k-1} a_g\left(\frac{n^2|D|}{d^2}\right) q^n.$$

5. Applications to central L-values

How would one discover the "right" decorations in the generating function $\Omega_{k,N}$? You may have noticed that the formula for $\Omega_{k,N}$ in terms of the $P_{k,N,m}$'s looks similar to the "classical" formulation of the Shimura lift. The reason for this is that the Poincaré series $P_{k,N,m}$ "pick off" Fourier coefficients of cusp forms.

Theorem (Petersson coefficient formula)**.** *For any $f \in S_k(N)$, we have*

$$\langle f, P_{k,N,m}\rangle = i_{4N}^{-1} \frac{\Gamma(k-\frac{1}{2})}{(4\pi m)^{k-\frac{1}{2}}} a_g(m).$$

That is, the Poincaré series $P_{k,N,m}$ is (up to a fixed constant) the **kernel function** for the m-th Fourier coefficient on $S_k(N)$ with respect to the Petersson inner product.

Sketch of proof. This is a classical technique called the "unfolding trick." Let's do it for $f \in S_{2k}$, that is, in the case of level 1 and even weight. Denote the seed of the Poincaré series by $\varphi_m(\tau) := q^m$. Then we find that

$$\langle f, P_{2k,m}\rangle = \int_{\Gamma(1)\backslash\mathbb{H}} f(\tau) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)} \overline{\varphi_m(\gamma\tau)}(c\overline{\tau}+d)^{-2k} v^{2k-2} du dv$$

$$= \sum_{\Gamma_\infty\backslash\Gamma(1)} \int_{\mathcal{F}} f(\gamma\tau)\varphi_m(\gamma\tau)\,\mathrm{Im}(\gamma\tau)^{2k}\frac{dudv}{v^2}$$

$$= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)} \int_{\gamma\cdot\mathcal{F}} f(\tau)\varphi_m(\tau)v^{2k-2}dudv$$

$$= \int_{\Gamma_\infty\backslash\mathbb{H}} f(\tau)\overline{\varphi_m(\tau)}v^{2k-2}dudv.$$

Now a fundamental domain for $\Gamma_\infty\backslash\mathbb{H}$ is very simple! Its just a vertical strip $0<u<1$, $v>0$. Thus, we obtain

$$\int_0^\infty \int_0^1 f(\tau)\overline{\varphi_m(\tau)}v^{2k-2}dudv.$$

As in the formula we've written before for writing Fourier coefficients of functions as integrals over horizontal lines of length one, the integral over u picks off the m-th Fourier coefficient, and the integral over v just gives the other fixed constant. $\square$

This is also used to show the main result we are after here.

Theorem (Kohnen-Zagier, Kohnen)**.** *Let D be a discriminant with $(-1)^kD>0$ and for all primes $p|N$, we have $\left(\frac{D}{p}\right)=W_p$, the Atkin-Lehner eigenvalue of $f\in S_{2k}^{\mathrm{new}}(N)$ a weight $2k$ newform. Then for square-free levels N, there is an Atkin-Lehner-Li-style newform theory for $S^+_{k+\frac{1}{2}}(4N)$, the* **Kohnen plus space** *of forms whose n-th Fourier coefficients vanishes unless $(-1)^kn\equiv 0,1 \pmod 4$.*

Then, **as Hecke modules***, we have an isomorphism*

$$S^{\mathrm{new},+}_{k+\frac{1}{2}}(4N)\cong S^{\mathrm{new}}_{2k}(N).$$

In particular, $\mathcal{S}_{k,N,D}$ preserves newforms and commutes with all Hecke operators.

Moreover, if $g=\sum_n c(n)q^n\in S^{\mathrm{new},+}_{k+\frac{1}{2}}(4N)$ corresonds to a weight $2k$ level N newform f, then we have

$$\frac{|c(|D|)|^2}{\langle g,g\rangle}=2^{\nu(N)}\frac{(k-1)!}{\pi^k}|D|^{k-\frac{1}{2}}\frac{L(f\otimes\chi_D,k)}{\langle f,f\rangle}.$$

Here, $\nu(N)$ denotes the number of prime divisors of N. In particular,

$$|c(|D|)|^2 \doteq L(f\otimes\chi_D,k),$$

and so

$$c(|D|)=0 \iff L(f\otimes\chi_D,k)=0.$$

This is a more explicit version in our case of a more general earlier result of Waldspurger.

6. Tunnell's Theorem

Tunnell applied this relationship between coefficients of the corresponding half-integral weight modular form and central L-values of quadratic twists of the integral weight modular form to study the Congruent Number Problem. To do this, he took the congruent number elliptic curve E_1, which has level 32. Let $f \in S_2(32)$ be the unique normalized newform (this happens to be a one-dimensional space), which thus corresponds to E_1 under the Modularity Theorem. We saw before that, under BSD,

$$n \text{ is congruent } \iff L(E_n, 1) = 0 \iff L(f \otimes \chi_n, 1) = 0.$$

Combining with the above, we expect for g a weight 3/2 level 128 modular form corresponding to f under the Shimura/Shintani correspondence, that

$$n \text{ is congruent } \iff a_g(n) = 0.$$

The fact that the level is 32, so not square-free or even cube-free, as well as that it has a lot of 2's makes things annoying, so one cannot apply Kohnen-Zagier immediately. But it can be made to work. Tunnell did such a calculation, and while we won't give all the details, we can note that such a calculation is a finite one as the modular forms spaces involved are finite-dimensional (and not even very large dimensions).

Theorem (Tunnell). *There are explicit modular forms f, f' in $S_{\frac{3}{2}}(128)$ and $S_{\frac{3}{2}}(128, \chi_2)$, respectively, which correspond to $f \in S_2(32)^{\text{new}}$ under the Shimura/Shintani correspondence. For the congruent number curves E_n, we then have the following Waldspurger-Kohnen-Zagier-type result:*

$$L(E_n, 1) \doteq \begin{cases} |a_f(n)|^2 & \text{if } n \text{ is odd}, \\ |a_{f'}(n)|^2 & \text{if } n \text{ is even}. \end{cases}$$

Here is roughly how Tunnell's calculation goes. To build up the weight 3/2 spaces, we use the nice theta function of weight 1 for the two-dimensional lattice $\mathbb{Z}^2$, which is also an eta product:

$$f_1(\tau) := \sum_{m,n \in \mathbb{Z}} (-1)^n q^{(4m+1)^2 + 8n^2} = \eta(8\tau)\eta(16\tau).$$

The representation in terms of eta functions in particular shows that this is a cusp form. Using the ordinary Jacobi theta function $\vartheta(\tau)$, we can get modular forms in $S_{\frac{3}{2}}(128, \chi_2)$ by forming the following products of f_1 with ϑ hit with V-operators:

$$f_1(\tau)\vartheta(2\tau), f_1(\tau)\vartheta(4\tau), f_1(\tau)\vartheta(8\tau), f_1(\tau)\vartheta(16\tau).$$

The products here are then sums over $\mathbb{Z}^3$ with ternary quadratic forms as powers of q when you write them out.

This is the last ingredient we need for Tunnell's main result if we wish to just assume BSD. But one direction of the above if and only if implications is actually known. This was first done by **Coates-Wiles**, who showed that if the rank of a CM elliptic curve is positive, then $L(E) = 0$. Although the congruent number curve has CM, so this would be sufficient for our purposes, its worth mentioning the best known progress towards BSD. Kolyvagin later showed that the Coates-Wiles result holds for any rational elliptic curve, even non-CM ones. Gross-Zagier and Kolyvagin in fact proved the following incredible result.

Theorem (Gross-Zagier, Kolyvagin)**.** *If a rational elliptic curve E has order of vanishing $\mathrm{ord}_{s=1}L(E,s)$ equal to 0 or 1, then*

$$\mathrm{rk}_{\mathbb{Q}}(E) = \mathrm{ord}_{s=1}L(E,s).$$

We finally have all the pieces needed, which we have been building to throughout the whole semester. Tunnell used the aforementioned explicit calculations to prove the following stunning result.

Theorem (Tunnell (1983))**.** *Let n be square-free, odd, and congruent. Then we have*

$$\#\left\{x,y,z \in \mathbb{Z} \mid n = 2x^2 + y^2 + 32z^2\right\} = \frac{1}{2}\#\left\{x,y,z \in \mathbb{Z} \mid n = 2x^2 + y^2 + 8z^2\right\}.$$

Let n be square-free, even and congruent. Then we have

$$\#\left\{x,y,z \in \mathbb{Z} \mid \frac{n}{2} = 4x^2 + y^2 + 32z^2\right\} = \frac{1}{2}\#\left\{x,y,z \in \mathbb{Z} \mid \frac{n}{2} = 4x^2 + y^2 + 8z^2\right\}.$$

In both cases, the converse is true assuming BSD.

This gives a surprising, and remarkably efficient way to check if n is congruent. In fact, Bach and Ryan showed that this criterion can be checked using time and space $O(n^{\frac{1}{2}+o(1)})$. Finding the explicit triangles that this criterion says exist if n passes the test is much harder, but other methods can be used to do so as well, as in Zagier's famous $n = 157$ example we saw at the beginning of the class.