THETA FUNCTIONS

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> La vie n'est bonne qu'à deux choses: à faire des mathématiques et à les professer. (The only two good things in life are doing mathematics and teaching it.)
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> Poisson, quoted by François Arago in Notices biographiques, Volume 2, 1854, p. 662.

1. First Example

We have seen one way to construct modular forms: as averages of slash operators. There is one other common way to build examples, which gives theta functions. Historically, these were the first examples of modular forms, due to their connections both to physics and combinatorics.

Our basic **prototype** is

$$\vartheta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + \ldots.$$

This has the following transformation properties.

Theorem. *For all $\tau \in \mathbb{H}$, we have*

$$\vartheta(\tau+1) = \vartheta(\tau),$$
$$\vartheta\left(-\frac{1}{4\tau}\right) = \sqrt{\frac{2\tau}{i}}\vartheta(\tau).$$

Proof. Translation invariance is clear as its a q-series. For the inversion property, we need **Poisson summation**. The key point will then be that Gaussian functions are essentially their own Fourier transform. This sort of observation is the key point behind any theta function construction.

So let's describe Poisson summation. In its most basic form, let $f\colon \mathbb{R} \to \mathbb{C}$ be a "nice" function. Here, we'll take it to be a Schwartz function (it and its derivatives all have rapid decay). You may have seen in analysis class that the nice thing about this class of functions is that it is closed under taking the Fourier transform. For really fast decaying things, like Gaussians, this will be good enough. Then Poisson summation says the following:

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n),$$

where $\widehat{f}(z)$ is the Fourier transform

$$\widehat{f}(z) := \int_{-\infty}^{\infty} e(-\tau z) f(\tau) d\tau.$$

Why does this work? Well, as long as f has suitable decay, you can form the function

$$g(\tau) := \sum_{n\in\mathbb{Z}} f(n+\tau).$$

This is visibly 1-periodic, and so has an expansion

$$g(\tau) = \sum c_n q^n,$$

where the Fourier coefficients are given by

$$c_m = \int_0^1 q^{-m} g(\tau) d\tau = \int_0^1 q^{-m} \sum_{n\in\mathbb{Z}} f(n+\tau) d\tau = \sum_{n\in\mathbb{Z}} \int_0^1 q^{-m} f(n+\tau) d\tau$$

$$\sum_{n\in\mathbb{Z}} \int_n^{n+1} q^{-m} f(\tau) d\tau = \int_{\mathbb{R}} q^{-m} f(\tau) d\tau = \widehat{f}(m).$$

Thus,

$$\sum_n f(n) = g(0) = \sum_n c_n = \sum_n \widehat{f}(n).$$

Intuitively, as long as you have this kind of decay, you can look at functions on $\mathbb{R}^d$ as well, and sums over a lattice are equal up to a constant to the sum of values of the Fourier transform over the "dual lattice".

Returning to $\vartheta(\tau)$, by the Identity Theorem, it suffices to check it holds on the imaginary axis. So let's set $\tau = it/2$, so that

$$-\frac{1}{4\tau} = \frac{i}{2t}$$

and

$$q = e(\tau) = e^{-\pi t} \implies q^{n^2} = e^{-\pi n^2}, \quad q^{n^2}\big|_{-1/4\tau} = e^{-\frac{\pi n^2}{t}}.$$

Letting $f(x) := e^{-\pi t x^2}$, for $t > 0$, we have

$$\widehat{f}(y) = \int_{\mathbb{R}} e^{-\pi t x^2 + 2ixy} dx = \frac{e^{-\pi y^2 t}}{\sqrt{t}} \int_{\mathbb{R}} e^{-\pi u^2} du = \frac{e^{-\frac{\pi y^2}{t}}}{\sqrt{t}},$$

where we have made the substitution $u = \sqrt{t}(x - iy/t)$ and shifted back the path of integration. Thus,

$$f(n) = q^{n^2}\big|_{\tau = it/2}, \quad \widehat{f}(n) = \frac{1}{\sqrt{t}} e^{-\frac{\pi n^2}{t}}.$$

Thus, Poisson summation gives

$$\vartheta(\tau) = \vartheta(it/2) = \sum_{n\in\mathbb{Z}} q^{n^2}\big|_{\tau=it/2} = \sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)$$

$$= \frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}} e^{-\frac{\pi n^2}{t}} = \frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}} q^{n^2}\big|_{\tau'=-1/4\tau} = t^{-\frac{1}{2}} \vartheta(-1/4\tau).$$

$\square$

2. Generalizations

The above construction can be generalized in many ways.

(1) The Gaussian can be replaced by similar functions which are essentially eigenfunctions of the Fourier transform. This leads to functions like the completions of Ramanujan's mock theta functions that Zwegers famously studied. That led to the birth of the field of harmonic Maass forms, now a very active field of research.
(2) You can twist this by roots of unity. This is really saying you can insert a second parameter which induces more transformation properties. This goes back to Jacobi's original work, and is closely tied to the heat equation from physics.
(3) You can do all this for higher dimensional lattices.

3. Congruence subgroup for $\vartheta(\tau)$

Where does $\vartheta(\tau)$ live? It is a modular form of weight 1/2 on

$$\langle T, \left(\begin{smallmatrix} 0 & -1 \\ 4 & 0 \end{smallmatrix}\right)\rangle \leq \Gamma(1).$$

The second matrix is a **Fricke involution**. In general level N, the Fricke involution is

$$W_N := \frac{1}{\sqrt{N}} \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right).$$

The subgroup of $\Gamma(1)$ generated by $\Gamma_0(N)$ and the Fricke involution has a special name:

$$\Gamma_0(N)^+ := \langle \Gamma_0(N), W_N \rangle .$$

There is a special fact for $N = 4$.

Proposition. *We have*

$$\langle T, W_4 \rangle = \Gamma_0^+(4).$$

Proof. Its easy to check that $W_N^2 = -1$, that is, that its actually an involution. Denote the conjugation $W_4 T W_4^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right)$ by $\widetilde{T}$. We will show that $T, \widetilde{T}$ generate $\overline{\Gamma_0(4)}$ (this "projectivization" of $\Gamma_0(4)$ is just what we get by modding out by $\{\pm 1\}$). Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$. Then a is odd, so trivially, $|a| \neq 2|b|$. If $|a| < 2|b|$, then $|b+a|$ or or $|b-a|$ is less than $|b|$. So letting $\gamma \mapsto \gamma T^{\pm 1}$ decreases a^2+b^2. If $|a| > 2b \neq 0$, then one of $a \pm 4b$ is smaller in absolute value than a, so $\gamma \mapsto \gamma \widetilde{T}^{\pm 1}$ decreases $a^2 + b^2$. Thus, multiplying on the right by T's and $\widetilde{T}$'s eventually gives $b = 0$. This resulting matrix is then of the form $\pm \widetilde{T}^n$ for some n. □

Thus, we've shown that

$$\vartheta(\tau) \in M_{\frac{1}{2}}(\Gamma_0(4)^+, \varepsilon_\vartheta),$$

where ε_ϑ is some multiplier system. I mentioned before that not too many of these exist; in fact, its related to the multiplier of η^3. All of the above computations were nicer since we used the Fricke involution, as this is almost like the inversion matrix S; nicer than if we used another generator of $\Gamma_0(4)$.

4. Combinatorial consequences

We now come to some **applications** of this modularity. Let $r_k(n)$ be the number of ways to represent n as a sum of k squares of integers. Then we see directly that

$$\sum_{n\geq 0} r_k(n)q^n = \vartheta^k(\tau).$$

Sums of 1 square are kind of boring, so let's consider some examples with $k > 1$.

4.1. **The case $k = 2$.** Fermat famously studied sums of two squares. These are dictated by ϑ^2. This is a form of weight 1, and with Nebentypus. The character is

$$\chi_{-4}(n) = \left(\frac{-4}{n}\right) = \begin{cases} +1 & \text{if } n \equiv 1 \pmod 4, \\ -1 & \text{if } n \equiv 3 \pmod 4, \\ 0 & \text{else.} \end{cases}$$

Now it turns out that $\dim(M_1(4, \chi_{-4})) = 1$ (dimension formulas like we proved for $\Gamma(1)$ exist in great generality). How can we find another representation of our form in this one dimensional space? We can use **Eisenstein series**. We can modify the weight k slash action to make invariance of a function f read as

$$f(\gamma\tau) = (c\tau + d)^k \chi(d) f(\tau).$$

Averaging up slash operators of the function 1 again gives the weight k character χ Eisenstein series, which have a very similar Fourier expansion:

$$G_{k,\chi}(\tau) = c_k(\chi) + \sum_{n\geq 1} \left(\sum_{d|n} \chi(d) d^{k-1} \right) q^n \in M_k(N, \chi),$$

where χ is any Dirichlet character mod N and

$$c_k(\chi) = \frac{1}{2} L(1-k, \chi)$$

is half a special value of the L-function for χ (the analytic continuation of the Dirichlet series $\sum_{n\geq 1} \chi(n) n^{-s}$).

In our case,

$$G_{1,\chi_{-4}}(\tau) = \frac{1}{2} L(0, \chi_{-4}) + \sum_{n\geq 1} \left(\sum_{d|n} \chi_{-4}(d) \right) q^n = \frac{1}{4} + q + q^2 + q^4 + 2q^5 + \ldots.$$

Here, I used the functional equation of $L(s, \chi_{-4})$ (similar to that for the Riemann zeta function which we worked out) to relate $L(0, \chi_{-4})$ to the value $L(1, \chi_{-4})$. This was evaluated by Leibniz:

$$L(1, \chi_{-4}) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}.$$

Thus, we find by comparing the leading Fourier coefficient that

$$\vartheta^2 = 4G_{1,\chi_{-4}}.$$

In particular, we see that for $n \geq 1$, we have

$$r_2(n) = 4\sum_{d|n} \chi_{-4}(d) = 4\sum_{\substack{d|n \\ d \text{ odd}}} (-1)^{\frac{d-1}{2}}.$$

Corollary (Fermat's Theorem on primes $p = \square + \square$)**.** *If $p \equiv 1 \pmod 4$, then p is a sum of two squares.*

Proof. We compute

$$r_2(p) = 4\left(1 + (-1)^{\frac{p-1}{2}}\right) = 8 > 0.$$

$\square$

Remark. *Note that the formula immediately implies that primes $p \equiv 3 \pmod 4$ are not sums of two squares, though this is obvious from the "congruence obstruction" that squares are always* 0 *or* 1 *mod* 4*.*

Of course, our formula gives a lot more information than Fermat's original result.

4.2. **The case $k = 3$.** The function ϑ^3 encodes the numbers $r_3(n)$. Proving this case is just a tiny bit beyond what we can do in this class, but is more like the next topic we could do if we had a second semester on the topic. Let $h(d)$ be the **class number** of $\mathbb{Q}(\sqrt{d})$. We will say more about these later. But the point is its a finite number which measures the obstruction from the algebraic integers inside $\mathbb{Q}(\sqrt{d})$ from being a UFD (they are a UFD iff $h(d) = 1$).

Theorem (Gauss)**.** *If $n > 3$ is square-free, then*

$$r_3(n) = \begin{cases} 12h(-n) & \text{if } n \equiv 1, 2 \pmod 4, \\ 24h(-n) & \text{if } n \equiv 3 \pmod 8) \\ 0 & \text{if } n \equiv 7 \pmod 8). \end{cases}$$

This is closely connected to Zagier's **non-holomorphic weight 3/2 Eisenstein series**. This is a prototypical harmonic Maass form. The point is that the generating function for $h(-d)$ is nearly modular of weight 3/2, but it only works if you add an extra non-holomorphic function to make it modular.

4.3. **The case $k = 4$.** As we've already seen, even powers are easier, as the modular forms spaces involved are easier. Let's look at sums of 4 squares. We then want to study $\vartheta^4 \in M_2(4)$. Multiples of 4 are thus even easier as the character and the multiplier become trivial. Now $M_2(4)$ two-dimensional space. We can find a basis for this space as follows. Recall that

$$G_2^* = G_2 + \frac{1}{8\pi v}$$

transforms as a level 1 modular form of weight 2. The non-holomorphic pieces (with $1/v$) cancel out in the following:

$$G_2^*(\tau) - NG_2^*(N\tau) = G_2(\tau) - NG_2(N\tau).$$

Thus, this is a holomorphic modular form of weight 2. Letting $\tau \mapsto N\tau$ (this is applying V_N) raises the level to $\Gamma_0(N)$. Thus,

$$G_2(\tau) - NG_2(N\tau) \in M_2(N).$$

In our case, we can obtain a basis for $M_2(4)$ as

$$\{G_2(\tau) - 2G_2(2\tau), G_2(2\tau) - 2G_2(4\tau)\}.$$

Note that the second function is the first hit with V_2, but that this didn't actually raise the level! Thus, to write ϑ^4 in this basis, we need the first two coefficients:

$$\vartheta^4 = 1 + 8q + \ldots \implies \vartheta^4 = 8(G_2(\tau) - 2G_2(2\tau)) + 16(G_2(2\tau) - G_2(4\tau)).$$

Reading off coefficients gives

$$r_4(n) = 8 \sum_{\substack{d|n \\ 4\nmid d}} d.$$

Corollary (Lagrange's Theorem on sums of four squares)**.** *Every positive integer is a sum of four squares.*

Proof. The number 1 is always a divisor of n not divisible by 4. $\square$

4.4. **Higher k.** For higher powers still, its possible to give infinite families of identities relating powers of ϑ to Eisenstein series via special continued fractions, by work of Milne. The amazing fact is that you only need about the square root of the dimension different "inputs", which is about the square root of what you'd expect. This includes other classical identities.

Example. *We have*

$$r_8(n) = 16 \sum_{d|n} (-1)^{n+d} d^3.$$

There are also nice classical formulas for $\vartheta(-q)^{16}$ and $\vartheta(-q)^{24}$, due to Glaisher and Ramanujan. Twisting by ± 1 can be useful.

5. Generalizing to Jacobi forms

This brings us to our important generalization, which Jacobi thoroughly investigated:

$$\vartheta(z;\tau) := \sum_{n \in \frac{1}{2} + \mathbb{Z}} (-1)^n q^{\frac{n^2}{2}} \zeta^n.$$

This is a **Jacobi form**, which means that it has a modular transformation equation, but also an "elliptic" transformation equation under shifting z by elements of the lattice

$\mathbb{Z}\tau+\mathbb{Z}$ (instead of being an elliptic function in the second variable, it "nearly" is). In Jacobi form parlance, invented by Eichler and Zagier, this has weight and index $1/2$, and multiplier the multiplier of η^3. Jacobi forms encode infinite families of modular forms in multiple ways. In particular, specialization to **torsion points** $z=\alpha\tau+\beta$, $\alpha,\beta\in\mathbb{Q}$ always gives a modular form of weight $1/2$ on some higher level. The β term allows you to twist by roots of unity (which by adding up can restrict sums to arithmetic progressions for studying congruences!), and the α term adds a linear term in the exponent of q.

Generalizing further, there are natural theta functions in higher dimensions. To describe this, let $Q\colon \mathbb{Z}^r\to\mathbb{Z}$ be a positive definite, integral **quadratic form** (homogenous polynomial of degree 2). Then the **bilinear form** attached to Q is $B(x,y):=Q(x+y)-Q(x)-Q(y)$.

Theorem (Hecke-Schönberg, Eichler-Zagier)**.** *For any $a\in\mathbb{Z}^r$, the theta function*

$$\sum_{n\in\mathbb{Z}^r} q^{Q(n)}\zeta^{B(n,a)} \quad (\zeta := e(z))$$

is a Jacobi form of weight $r/2$ and index $Q(a)$ on a congruence subgroup.

6. Further generalizations

What's also allowed more generally is you can take other functions that are essentially eigenfunctions of the Fourier transform.

6.1. General picture à la Vignéras.

This can be written down explicitly as giving a special differential equation of Vignéras. These solutions lead to Hermite functions and error functions, the latter of which leads to Zwegers' discovery on Ramanujan's mock theta functions. The simplest functions satisfying the Vignéras equation are ones which are eigenfunctions of the **Euler operator**

$$\sum x_i\frac{\partial}{\partial x_i}$$

and which are in the kernel of the **Laplacian** Δ_Q. The Euler operator condition just says that the function is homogenous, and the Laplacian is a rescaled one of the usual Euclidean one. Specifically, if $\langle\cdot,\cdot\rangle$ is the inner product on $\mathbb{R}^r$, then $Q(x)=\frac{1}{2}\langle x,Ax\rangle$ for an associated **symmetric matrix** A, and then

$$\Delta_Q := \left\langle \frac{\partial}{\partial x}, A^{-1}\frac{\partial}{\partial x}\right\rangle.$$

6.2. Sphereical polyomials.

The nicest subcase of these are the **spherical polynomials**. These are homogenous polynomials of degree d, which are harmonic (killed by Δ_Q). For such a polynomial P, this leads to the theta function

$$\theta_{Q,P}(\tau)=\sum_{n\in\mathbb{Z}^r} P(n)q^{Q(n)},$$

which is modular of weight $\frac{r}{2}+d$.

Finally, the nicest non-trivial case of **this** is the spherical polynomial $P(n)=n$. This together with twists by Dirichlet characters χ leads to **Shimura theta functions**

$$\vartheta_\chi(\tau)=\sum_{n\in\mathbb{Z}} n\chi(n)q^{n^2}$$

of weight 3/2 (we only want to consider this for *odd* characters $\chi(-n)=\chi(n)$, or else the sum is zero anyways). These are some of the most famous theta functions, and very useful for building up examples of weight 3/2 modular forms. We'll need weight 3/2 theta functions, for example, when we return to the Congruent Number Problem (though this will use weight 3/2 theta functions which rather come from quadratic forms in 3 dimensions).