MODULAR FORMS LECTURE 21: L-FUNCTIONS FOR MODULAR FORMS

LARRY ROLEN, VANDERBILT UNIVERSITY, FALL 2020

> When the Zetas fill the sky, Will our leaders tell us why? Fully loaded satellites, Will conquer nothing but our minds.
>
> — Matthew Bellamy

Given $\sum_{n\geq 0} a_n q^n = f \in S_k$, we define the **$L$-function** attached to f as the **Dirichlet series**

$$L(f,s) := \sum_{n\geq 1} \frac{a_n}{n^s} \qquad \mathrm{Re}(s) \gg 0.$$

The nicest case is when f is a Hecke eigenform. Then the coefficients are multiplicative, and the L-function has an **Euler product**. To see this, let's look at the prototypical example.

Example. *The* **Riemann zeta function** *is*

$$\zeta(s) := \sum_{n\geq 1} \frac{1}{n^s} \qquad (\mathrm{Re}(s) > 1).$$

Euler noticed that it factorizes as

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}},$$

where the product is over the set of prime numbers. This can be though of as the "analytic version of the Fundamental Theorem of Arithmetic." Namely, it is just a generating-function encoding of the fact that all integers factor uniquely as products of primes.

The same type of Euler product, a product of similar terms over primes, holds for any Dirichlet series $\sum_{n\geq 1} b_n n^{-s}$ whenever the b_n are multiplicative. In our case, we have

$$L(f,s) = \prod_p \left(1 + \frac{a_p}{p^s} + \frac{a_{p^2}}{p^{2s}} + \dots\right).$$

This only used multiplicativity. But we also have the Hecke relations for prime powers too! Suppose that a_{p^r} are a sequence satisfying Hecke relations at prime powers, and consider the generating function

$$A_p := \sum_{r\geq 0} a_{p^r} x^r.$$

Then

$$A_p = 1 + \sum_{r\geq 0} a_{p^{r+1}} x^{r+1} = 1 + \sum_{r\geq 0} a_p a_{p^r} x^{r+1} - \sum_{r\geq 1} p^{k-1} a_{p^{r-1}} x^{r+1} = 1 + a_p x A_p - p^{k-1} x^2 A_p.$$

Solving yields

$$A_p = \frac{1}{1 - a_p x + p^{k-1} x^2}.$$

Letting $x = p^{-s}$ and plugging in our previous formula gives.

Proposition. *If f is a Hecke eigenform in S_k, then $L(f, s)$ has the following Euler product:*

$$L(f, s) = \prod_p \frac{1}{1 - a_p p^{-s} + p^{k-1-2s}}.$$

Example. *We can also define L-functions for non-cusp forms. The term $n = 0$ in the Dirichlet series won't make sense, so we just ignore it and still define for $f \in M_k$*

$$L(f, s) := \sum_{n\geq 1} \frac{a_f(n)}{n^s}.$$

We've seen that the coefficients of Eisenstein series involve zeta values, so it will not be a surprise to find that $L(G_k, s)$ is related to $\zeta(s)$. Specifically, we work with the Euler product computation above and similarly look at

$$a_{p^r} = \sigma_{k-1}(p^r) = \frac{p^{(r+1)(k-1)} - 1}{p^{k-1} - 1},$$

where here we used the geometric series. Then as above, we compute

$$A_p = \sum_{r\geq 0} \frac{p^{(r+1)(k-1)} - 1}{p^{k-1} - 1} x^r = \frac{1}{(1 - p^{k-1}x)(1 - x)}.$$

Thus, we obtain

$$L(G_k, s) = \prod_p \frac{1}{1 - \sigma_{k-1}(p)p^{-s} + p^{k-1-2s}} = \prod_p \frac{1}{1 - p^{k-1-2s}} \cdot \prod_p \frac{1}{1 - p^{-s}} = \zeta(s-k+1)\zeta(s).$$

We now look at the region of convergence of our definitions of modular L-functions. When Dirichlet series converge absolutely, they do so in a right half-plane. This is because $|n^{-s}|$ depends only on $\mathrm{Re}(s)$. The region depends on the growth rate of the coefficients a_n. We've shown that for a cusp form, the coefficients satisfy the asymptotic bound $a_n = O(n^{\frac{k}{2}})$ (which can be improved by the work of Deligne). This is enough to guarantee that $L(f,s)$ converges for $\mathrm{Re}(s) > \frac{k}{2}+1$.

As Riemann showed holds for $\zeta(s)$, Hecke showed that these L-functions have analytic continuations to the whole complex plane, and have a functional equation. These follow from properties of the **Mellin transform**. Recall that the Gamma function is

$$\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}dt, \qquad (\mathrm{Re}(s) > 0).$$

This extends the factorial function via the relation $\Gamma(n+1) = n!$ for $n \in \mathbb{Z}_{\geq 0}$. This is because just as for $n!$, we have the relation $\Gamma(s+1) = s\Gamma(s)$. This gives a continuation of $\Gamma(s)$ to a meromorphic function on $\mathbb{C}$ with poles at $\mathbb{Z}_{\leq 0}$. Similarly, for all ζ and L-functions, the analytic continuation will come from a functional equation.

The Gamma function is the prototypical Mellin transform. More generally, if $\varphi(t)$ is a real-function defined for $t > 0$ and it decays rapidly at ∞ (faster than any power of t) and blows up at most like a polynomial near 0 (is $O(t^{-C})$ as $t \to 0$) , then the **Mellin transform** is

$$\mathcal{M}(\varphi)(s) := \int_0^\infty \varphi(t)t^{s-1}dt.$$

This converges absolutely and locally uniformly for $\mathrm{Re}(s) > C$.

A special case is when

$$\varphi(t) = \sum_{n\geq 1} c_n e^{-nt},$$

where the c_n are growing polynomially. Then we find

$$\mathcal{M}(\varphi)(s) = \int_0^\infty \sum_{n\geq 1} c_n e^{-nt}t^{s-1}dt = \int_0^\infty e^{-t}t^{s-1}dt \sum_{n\geq 1} c_n n^{-s} = \Gamma(s)\sum_{n\geq 1} c_n n^{-s}.$$

That is, the Mellin transform is $\Gamma(s)$ times the Dirichlet series for $\{c_n\}$.

We also need the following **general principle**. Suppose that

$$\varphi\left(\frac{1}{t}\right) = \sum_{j=1}^{J} A_j t^{\lambda_j} + t^h\varphi(t), \qquad h, A_j, \lambda_j \in \mathbb{C}.$$

Then by writing the Mellin transform as $\int_0^\infty = \int_0^1 + \int_1^\infty$ and letting $t \mapsto 1/t$ in the first integral gives (for $\mathrm{Re}(s) \gg 0$)

$$\mathcal{M}(\varphi)(s) = \int_1^\infty \left(\sum_{j=1}^{J} A_j t^{\lambda_j} + t^h\varphi(t)\right) t^{-s-1}dt + \int_1^\infty \varphi(t)t^{s-1}dt = \sum_{j=1}^{J} \frac{A_j}{s-\lambda_j} + \int_1^\infty \varphi(t)\left(t^s + t^{h-s}\right)\frac{dt}{t}.$$

The integral $\int_1^\infty$ is clearly invariant under $s \mapsto h-s$. For the sum in the last display, apply the expansion formula of $\varphi(1/t)$ twice to find that for each j, there is a j' with $\lambda_{j'} = h - \lambda_j$, $A_j' = -A_j$.

This gives that $\mathcal{M}(\varphi)(s)$ analytically continues to a meromorphic function in $\mathbb{C}$ with poles at $s = \lambda_1, \lambda_2, \ldots, \lambda_J$.

Example. *In the case of the Riemann zeta function, let* $\varphi(t) = \sum_{n\geq 1} e^{-\pi n^2 t}$ *(this is basically a theta function minus its constant term). Then we have*

$$\mathcal{M}(\varphi)(s) = \pi^{-s}\Gamma(s)\zeta(2s),$$

and the general principle holds with $h = 1/2$, $J = 2$, $\lambda_1 = 0$, $\lambda_2 = 1/2$, $A_2 = -A_1 = 1/2$. *This follows by using the modularity of the theta function which we will shortly study (this is a theta function minus a constant term on the imaginary axis at* $\tau = it$*, and letting* $t \mapsto 1/t$ *corresponds to looking at the modular inversion symmetry). Rewriting the resulting functional equation gives the following.*

Theorem (Riemann)**.** *The* **completed** *zeta function* $\xi(s) := \frac{1}{2}\pi^{-\frac{s}{2}}s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s)$ *is entire and satisfied the functional equaion*

$$\xi(s) = \xi(1-s).$$

Alternatively, $\zeta(s)$ *has an analytic continuation to* $\mathbb{C}$ *with a simple pole just at* $s = 1$*, and with the functional equation*

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).$$

Example. *If* $f \in M_k$*, let* $\varphi(t) = f(it) - a_0 = \sum_{n\geq 1} a_n e^{-2\pi n t}$*. This is small at* ∞*, and we compute*

$$\varphi\left(\frac{1}{t}\right) = f\left(\frac{-1}{it}\right) - a_0 = (it)^k f(it) - a_0 = (-1)^{\frac{k}{2}} t^k \varphi(t) + (-1)^{\frac{k}{2}} a_0 t^k - a_0.$$

Then we have

$$\mathcal{M}(\varphi)(s) = (2\pi)^{-s}\Gamma(s)L(f,s),$$

and the above expansion implies that $L(f,s)$ *has an analytic continuation to* $\mathbb{C}$ *which is entire if* $f \in S_k$*, and has poles at* $s = 0, k$ *if* f *is non-cuspidal. Since this "completed" L-function has poles at* $0, k$*, the original L-function*

$$L(f,s) = (2\pi)^s \frac{\mathcal{M}(\varphi)(s)}{\Gamma(s)}$$

has a pole just at $s = k$ *(the pole of the completed L-function at* $s = 0$ *is cancelled by the simple zero at* $s = 0$ *of* $1/\Gamma(s)$*). Further, we have the functional equation*

$$(2\pi)^{-s}\Gamma(s)L(f,s) = (-1)^{\frac{k}{2}}(2\pi)^{s-k}\Gamma(k-s)L(f,k-s).$$