MODULAR FORMS LECTURE 20: EIGENFORM BASES

LARRY ROLEN, VANDERBILT UNIVERSITY, FALL 2020

> The Opera ghost really existed. He was not, as was long believed, a creature of the imagination of the artists, the superstition of the managers, or a product of the absurd and impressionable brains of the young ladies of the ballet, their mothers, the box-keepers, the cloak-room attendants or the concierge. Yes, he existed in flesh and blood, although he assumed the complete appearance of a real phantom; that is to say, of a **spectral** shade.
>
> Gaston Leroux, The Phantom of the Opera

Our main result today is the following.

Theorem. *For any k, there is a basis of M_k consisting of Hecke eigenforms.*

Proof. Since $M_k = \mathbb{C}\cdot E_k \oplus S_k$, and E_k is an eigenform, we just need to show S_k has a basis of eigenforms. This follows from **spectral theory**. For this, we need the **Petersson inner product** on S_k, defined by

$$(f, g) := \int_{\Gamma(1)\backslash\mathbb{H}} v^k f(\tau)\overline{g(\tau)} d\mu.$$

Here, $d\mu = dudv/v^2$ is the *hyperbolic metric*. Recall that we saw before that $v^k|f(\tau)|^2$ is $\Gamma(1)$-invariant. The same calculation shows more generally that $v^k f(\tau)\overline{g(\tau)}$ is too, and that so is the metric $d\mu$. So the integral on this quotient is well-defined. Now, a very quick calculation shows that the volume of the fundamental domain $\int_{\Gamma(1)\backslash\mathbb{H}} d\mu$ is finite, and the cuspidality of f and g ensures boundedness, so (f, g) converges absolutely. Note that this also works if one of f, g is in S_k, and the other is in M_k. It fails to converge if both are non-cusp forms; however, the integral can be regularized to make it work out (these types of regularizations are important in physics, and were pioneered by Harvey and Moore).

This inner product makes S_k into a **finite-dimensional Hilbert space**. We also have an infinite sequence of commuting operators acting on these spaces, T_n. These operators are also **self-adjoint**:

$$(f|T_n, g) = (f, g|T_n).$$

The Spectral Theorem then implies that S_k is spanned by simultaneous eigenforms, as desired. □

The self-adjointness also has other nice consequences. For instance, if f is a normalized eigenform with coefficients a_n, then

$$a_n(f,f) = (a_n f, f) = (\lambda_n f, f) = (f|T_n, f) = (f, f|T_n) = (f, \lambda_n f) = (f, a_n f) = \bar{a_n}(f,f).$$

Since $(f,f) > 0$, this implies that **the coefficients are real**.

Even better, we have the following.

Theorem. *The Fourier coefficients of a normalized eigenform in S_k are algebraic integers of degree less than or equal to* $\dim S_k$.

Proof. By building all modular forms out of E_4, E_6, we can find a basis of S_k with $\mathbb{Z}$-integral coefficients. By the formula for the action of T_n on Fourier expansions, T_n preserves the lattice L_k of such $\mathbb{Z}$-integral forms. Let $f_1, \ldots, f_d$ be a $\mathbb{Z}$-basis for the lattice. The action of T_n with respect to this basis is a $d \times d$ matrix with entries in $\mathbb{Z}$, so the eigenvalues of the matrix are algebraic integers of degree at most d. But these *are* the coefficients of the normalized eigenforms. □

Example. *The first two dimensional space of cusp forms is S_{24}. We have a $\mathbb{Z}$-integral basis $f_1 = \Delta^2 = q^2 + O(q^3)$, $f_2 = E_4^3\Delta = q + O(q^2)$. So the normalized eigenforms have the shape $g = f_2 + \lambda f_1$. We can solve for λ:*

$$a_g(2) = \lambda + a_f(2), \quad a_g(4) = \lambda a_{f_1}(4) + a_{f_2}(4) = 1080\lambda + 12831808.$$

Now using the Hecke relation for $4 = 2^2$, we get

$$a_g^2(2) = a_g(4) + 2^{23} \implies \lambda^2 + 312\lambda - 20736000 = 0.$$

Hence,

$$g = f_2 + (-156 \pm 12\sqrt{144169})\Delta^2.$$

Since there are exactly 2 Hecke eigenforms in the space, these are them. They have Fourier coefficients in $\mathbb{Q}(\sqrt{144169})$, which is indeed a quadratic field, but has a huge discriminant.