MODULAR FORMS LECTURE 2: ELLIPTIC FUNCTIONS

LARRY ROLEN, VANDERBILT UNIVERSITY, FALL 2020

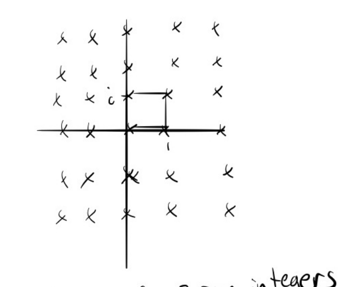
> Mankind is not a circle with a single center but an ellipse with two focal points of which facts are one and ideas the other.
>
> — Victor Hugo

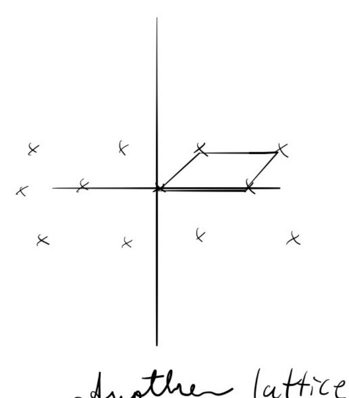
Before describing modular forms in more detail, we will discuss the related theory of elliptic functions. Modular forms as we will usually describe them are also called *elliptic modular forms* due to their connection to elliptic functions. In turn, this term comes from ellipses, specifically from earlier study of arc lengths of ellipses.

We start with a few basic definitions.

Definition. A **lattice** in $\mathbb{C}$ is a set of the form $\Lambda = \langle \omega_1, \omega_2 \rangle = \{m\omega_1 + n\omega)_2 : m, n \in \mathbb{Z}\}$, where $\omega_1, \omega_2 \neq 0$ and $\omega_1/\omega_2 \notin \mathbb{R}$.

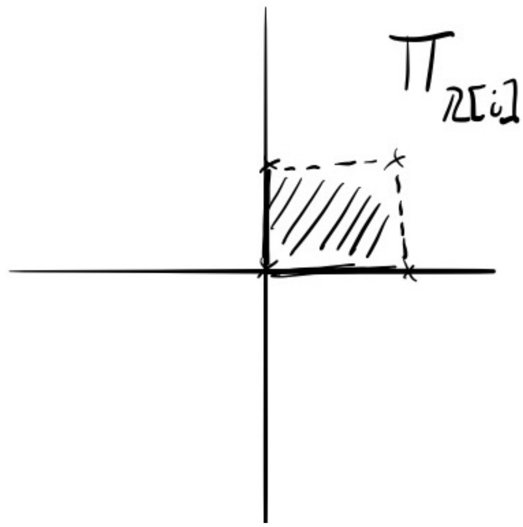
The condition that $\omega_1/\omega_2 \notin \mathbb{R}$ is needed as otherwise Λ would be contained in a line. This means that the quotient is a point in the lower or upper half plane, which is our first hint of why modular forms live on $\mathbb{H}$. Here are a few pictures.





Definition. The standard choice of **fundamental domain** for Λ is

$$\Pi = \Pi_\Lambda = \{\alpha\omega_1 + \beta\omega_2 : 0 \leq \alpha, \beta < 1\}.$$



The point is that every element of $\mathbb{C}$ is equivalent to exactly **one** point in Π **up to translation** by points in Λ. Some authors take a different definition with the dashed lines part, namely with whether points on the boundary of Π are included or not. It wouldn't then satisfy this uniqueness property, but it often doesn't matter if its only off on a measure 0 set. We can now make our key definitions.

Definition. A function $f : \mathbb{C} \to \mathbb{C}$ is **doubly periodic** (with respect to Λ) if

$$f(z + \lambda) = f(z)$$

for all $\lambda \in \Lambda$ and for all $z \in \mathbb{C}$.

Definition. A function on $\mathbb{C}$ is **elliptic** if it is meromorphic and doubly periodic. The set of all elliptic functions for Λ (really its a field) is denoted $\mathcal{E}_\Lambda$.

Remarks. (1) *If you haven't seen the word meromorphic before, here is a brief explanation. Holomorphic functions are nothing but the complex version of differentiable functions, except that now they are automatically smooth, and even analytic (which is very false for real-differentiable functions). Meromorphic functions are a generalization of holomorphic functions. They are the same, but they may have a discrete set of poles, points where the function blows up. At these points, it blows up like a rational function blows up. For example, any polynomial is a holomorhpic function, and any rational function is a meromorphic function, with poles where the denominator has a root. We say, for instance, that* $\frac{1}{(z-1)(z-2)^2}$ *is meromorphic with a* **simple** *pole at* $z = 1$ *and a* **double** *pole at* $z = 2$.

(2) *Even the meromorphic condition can be dropped. There are very nice examples which have slightly non-holomorphic functions, like a holomorphic function plus* $\mathrm{Im}(z)$*, which have had a lot of big applications in recent years related to BSD and moonshine.*

Exercise 1. *If you haven't taken complex analysis, look through the examples provided throughout these notes in detail and look up the topics mentioned. Of course, let me know if you have any questions! We will take a computational approach to this, and the complex analysis covered in this lecture is enough for us for the remainder of the semester.*

The following is used a lot in elliptic functions theory.

Key Result 1. *If* $f \in \mathcal{E}_\Lambda$ *has no pole in* Π*, then* $f \in \mathbb{C}$*. That is, any holomorphic elliptic function is a constant.*

Proof. The closure $\overline{\Pi}$ is compact. Thus, such an f is bounded on $\overline{\Pi}$. But all values of f on $\mathbb{C}$ are "repeats" of these values. Thus, f is bounded on $\mathbb{C}$. By Liouville's Theorem, f is a constant (if you haven't seen Liouville's Theorem, this is precisely its statement!). □

We also need the following.

Key Result 2. *Suppose* $f \in \mathcal{E}_\Lambda$ *has no poles on the boundary* $\partial\overline{\Pi}$ *(or, if need be, a slightly shifted parallelogram, like* $z_0 + \Pi$*). Then the sum of residues of* f *in* Π *is* 0*.*

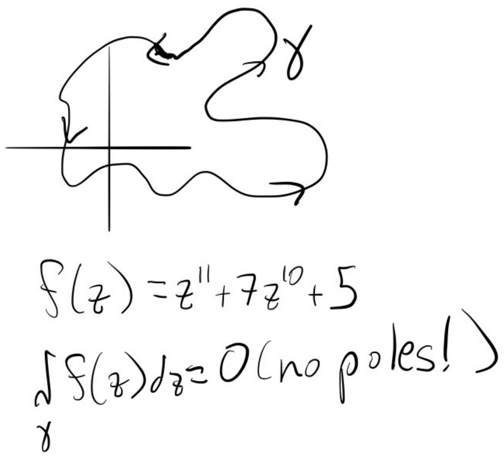
Proof. There is an important fact from complex analysis called the **Residue Theorem**. If a meromorphic function has a pole at $z = a$, then it has a **Laurent expansion** of the form

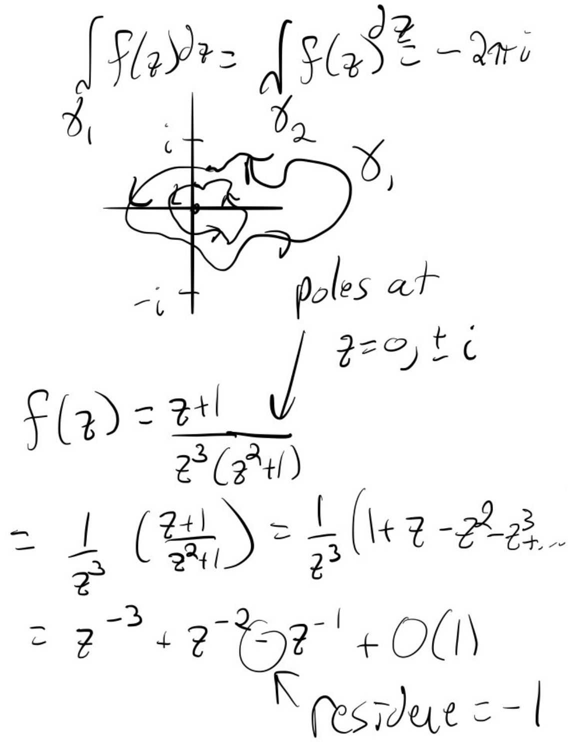
$$\sum_{n=-m}^{\infty} a_n (z-a)^m$$

around $z = a$, where m is the order of the pole. This is just like a Taylor series expansion, but some powers are negative. The **residue** is the value a_{-1}. The Residue Theorem states that for any meromorphic function and any simply closed curve traversed counterclockwise, we have

$$\sum_{a \text{ is a pole of } f \text{ on the inside of the curve}} \Big(\text{Residue of } f \text{ at } z = a\Big) = \frac{1}{2\pi i}\int_\gamma f(z)dz.$$

For instance, we have the following simple examples.



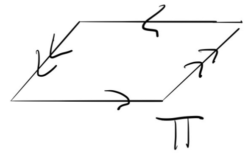


If you haven't seen the Residue Theorem, you should read about it, as its very useful for many situations (even real variable integrals), and we will be returning to it later.

Going back to the situation of elliptic functions, we integrate around the boundary of the fundamental domain. The residue theorem then says that

$$\sum(\text{residues inside }\Pi) = \frac{1}{2\pi i}\int_{\partial\Pi} f(z)dz.$$

The integral is zero. This is because the integrals over opposite sides cancel as the values are equal by double periodicity and the orientations are opposite (see the following picture). This completes the proof.



□

Corollary. *Non-constant elliptic functions have at least two poles (counting multiplicity) in a fundamental domain.*

Proof. We already saw that non-constant elliptic functions have at least one pole. If there is only one pole, counted with multiplicity, then this is a simple pole. But since the sum of residues is zero, the residue of this simple pole is 0. But then the Laurent expansion near that point is $0 \cdot (z-a)^{-1} + a_0 + \ldots = a_0 + O(z-a)$. Then there is no pole at $z = a$ after all. □

Remark. *It is possible for mildly non-holomorphic elliptic functions to have just* 1 *pole!*

Finally, we have the following.

Key Result 3. *If $f \in \mathcal{E}_\Lambda$ has no zeros or poles on $\partial\overline{\Pi}$, then*

$$\sum(\textit{ orders of zeros in } \Pi) = \sum(\textit{orders of poles in } \Pi).$$

Remark. *We have defined the order of a pole. If you haven't seen it, then the order of the zero at $z = a$ is as follows. If $f(a) = 0$, then it has a Taylor expansion near a of the shape $f(z) = \sum_{n \geq m} a_m (z-a)^m$. If $a_m \neq 0$, then we say that m is the order of the zero.*

Proof. We use the main idea behind what's known in complex analysis as the *argument principle*. The main observation is that the **log derivative**, f'/f, has **simple poles** wherever f has a zero or a pole, and the residues of the log derivative are equal to the order of the zero or pole. The residue is $+m$ if the zero is of order m, and the residue is $-m$ is there is a pole of order m. The derivative of an elliptic function for Λ is still in $\mathcal{E}_\Lambda$, and a quotient of elliptic functions is also elliptic. Thus, $f'/f \in \mathcal{E}_\Lambda$. Thus, by Key Result 2, and the observation we have

$$\sum(\text{orders of zeros of } f) - \sum(\text{orders of poles of } f) = \sum(\text{residues of } f) = 0.$$

This is equivalent to the claim. □

Remark. *There is a convenient term for zeros and poles which I may use in the future. The set of points where a function has either a zero or a pole is called its* **divisor**.

Exercise 2. *Check that the set $\mathcal{E}_\Lambda$ is closed under addition, multiplication, division, and quotients. In particular, $\mathcal{E}_\Lambda$ is a field, and $f'/f \in \mathcal{E}_\Lambda$ for any $f \in \mathcal{E}_\Lambda$.*

Exercise 3. *If you haven't seen it before, check the argument principle idea. That is, show that the log derivative has simple poles exactly at points in the divisor of f, and that the residues are the order or the zero or pole of f. As a hint, suppose that f has a zero of order m at a. Then write $f(z) = (z-a)^m g(z)$ for some function $g(z)$ with $g(a) \neq 0$. Now compute the residue and pole order of f'/f at a using this formula.*

In short, elliptic functions are natural extensions of periodic functions with strong conditions on their divisors. This makes comparing them easy. For instance, if I claim that two elliptic functions are equal, then all I have to do to check that is take the difference and show that the poles cancel out.

At this point, one could ask: Do natural elliptic functions exist? The answer is yes. Next time, we'll give some natural examples and describe some of their properties. After that, we will describe where modular forms come into the picture.

Exercise 4. *Can you define a non-trivial example of an elliptic function without looking it up? What would you try to define a doubly periodic function?*