MODULAR FORMS LECTURE 19: MORE ON HECKE OPERATORS

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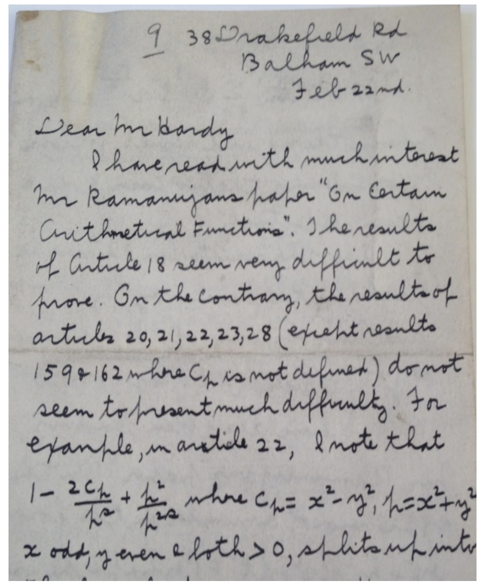
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Dear Mr Hardy
I have read with much interest Mr Ramanujans paper "On Certain Arithmetical Functions". The results of Article 18 seem very difficult to prove. On the contrary, the results of articles 20, 21, 22, 23, 28 (except results 159 & 162 where C_p is not defined) do not seem to present much difficulty. For example, in article 22, I note that
$1 - \frac{2c_p}{p^s} + \frac{p^2}{p^{2s}}$ where $c_p = x^2 - y^2$, $p = x^2 + y^2$
x odd, y even & both > 0, splits up into

Mordell's Letter to Hardy describing his work proving Ramanujan's conjecture on multiplicativity of $\tau(n)$ [Citation: Louis Joel Mordell's time in London by Ben Fairbairn, Birkbeck Pure Mathematics Preprint Series]

1. Modular forms as functions on lattices

We now make more explicit what we saw glimpses of when looking at elliptic functions. There is a natural bijection between modular forms and functions on lattices.

Definition. We say that lattices Λ, Λ' are **homothetic** if there is a $\lambda \in \mathbb{C}^\times$ such that $\Lambda' = \lambda\Lambda$. We say that a function F from the set of lattices to $\mathbb{C}$ is **homogenous of degree** $-k$ if $F(\lambda\Lambda) = \lambda^{-k}F(\Lambda)$ for all $\lambda \in \mathbb{C}^\times$.

Remark. *The following will show that functions homogeonous of weight* 0 *correspond to modular functions; the* j*-function then shows that two elliptic curves over* $\mathbb{C}$ *are isomorphic if and only if the corresponding lattices are homothetic.*

The connection to modular forms is given by the following calculation.

Exercise 1. *Show that for any lattice* Λ*, there is some* $\tau \in \mathbb{H}$ *such that* Λ *is homothetic to* $\mathbb{Z}\tau + \mathbb{Z}$*. Further show that* $\mathbb{Z}\tau' + \mathbb{Z}$ *is homothetic to* $\mathbb{Z}\tau + \mathbb{Z}$ *if and only if* $\tau' = \gamma\tau$ *for some* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$*.*

Based on these observations, we obtain a bijection

$$\{\text{modular forms of weight } k\} \leftrightarrow \{\text{functions on lattices homogenous of degree } -k\}.$$

Given F a function on lattices, the associated modular form is $f(\tau) = F(\langle \tau, 1\rangle)$. Given a modular form f, the corresponding function on lattices is $F(\langle \omega_1, \omega_2\rangle) = \omega_2^{-k} f(\omega_1/\omega_2)$, where ω_1, ω_2 are ordered so that $\omega_1/\omega_2 \in \mathbb{H}$.

2. Hecke operators on lattice functions

Given a homogenous function on lattices of degree -k, we define the n**-th Hecke operator** as a normalizing factor times a sum over values at sublattices of index n:

$$T_n(F)(\Lambda) := n^{k-1} \sum_{\substack{\Lambda' \subseteq \Lambda \\ [\Lambda \colon \Lambda']=n}} F(\Lambda').$$

It is easy to see that this is automatically homogenous of degree $-k$. Thus, the corresponding function on $\mathbb{H}$ is modular of weight k. Translating into modular forms language, this gives us the following (note that the power of n shifts due to the determinant factor in the slash operator):

$$f|T_n = n^{\frac{k}{2}-1} \sum_{\gamma \in \Gamma(1) \backslash M_n} f|\gamma.$$

Here M_n is the set of 2×2 integer matrices of determinant n. To see how this translation works, we note that below we'll show that a set of representatives of $\gamma \in \Gamma(1)\backslash M_n$ will be given by the set S_n of matrices $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with $ad = n$, $a, d > 0$, $0 \leq b < n$.

Exercise 2. *Given $M \in S_n$ and a lattice $\Lambda = \langle \omega_1, \omega_2 \rangle$, let Λ_M be the lattice $\langle a\omega_1 + b\omega_2, d\omega_2 \rangle$. Show that this map from S_n to sublattices of Λ is a bijection between S_n and sublattices of index n.*

To connect with our earlier definition, we compute the action of Hecke operators on Fourier expansions.

Theorem. *The following are true.*

(1) *If $f = \sum_{n\geq 0} a_n q^n \in M_k$, then $f|T_n = \sum_{m \geq 0} b_m q^m \in M_k$, where $b_k = \sum_{d|n,m} d^{k-1} a_{nm/d^2}$. In particular, T_k preserves the space of cusp forms.*

(2) *We have*

$$T_n T_m = \sum_{d|n,m} d^k T_{nm/d^2}.$$

In particular, $T_nT_m = T_mT_n$ for all m, n, and for coprime m, n we have $T_nT_m = T_{mn}$.

Proof. We first compute a set of representatives for $\Gamma(1)\backslash M_n$. If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \mu \in M_n$ with $c \neq 0$, then we can pick $\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$ with $\frac{a'}{c'} = \frac{a}{c}$ (if you reduce the fraction a/c into lowest terms, and want to find a matrix in $\Gamma(1)$ with the form $\left(\begin{smallmatrix} a & * \\ c & * \end{smallmatrix}\right)$, then you can solve for b and d to make the determinant 1 by Bezout as long as $(a, c) = 1$). With this choise,

$$\gamma^{-1}\mu = \left(\begin{smallmatrix} d' & -b' \\ -c' & a' \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} * & * \\ -ac'+a'c & * \end{smallmatrix}\right) = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in M_n.$$

Thus, we can choose all representatives in $\Gamma(1)\backslash M_n$ with the form $\mu = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$. Matrices in $\Gamma(1)$ of this form are precisely those in Γ_∞, which are those of the form $\gamma = \pm T^r$. Multiplying by one of these does the following:

$$\gamma \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = \pm \left(\begin{smallmatrix} a & b+dr \\ 0 & d \end{smallmatrix}\right).$$

Thus, we can choose one representative from each class by setting $a, d > 0$, $0 \leq b < d$.

Thus, writing in terms of slash operators,

$$f|T_n = n^{k-1} \sum_{\substack{a,d>0 \\ ad=n}} \sum_{b=0}^{d-1} d^{-k} f\left(\frac{a\tau + b}{d}\right).$$

The same formula we used before for exponential sums gives the claimed formula for the Fourier expansion.

Exercise 3. *Quickly check this using our earlier look at the action of such upper triangular matrices on Fourier expansions. Use this Fourier expansion formula to deduce part (2) of the theorem.*

□

Corollary. *If $n = p$ is prime, then*

$$f|T_p = \frac{1}{p} \sum_{j=0}^{p-1} f\left(\frac{\tau + j}{p}\right) + p^{k-1} f(p\tau) = f|U_p + p^{k-1} f|V_p.$$

Definition. We say $f \in M_k$ is a **Hecke eigenform** if $T_n f = \lambda_n f$ for some $\lambda_n \in \mathbb{C}$ for all n. We saw that f is a **normalized eigenform** if $a_f(1) = 1$.

As mentioned earlier for S_{12}, in any one dimensional space, a form is automatically an eigenform. For instance, the first few Eisenstein series are (though we've seen that all Eisenstein series are), and any cusp form of weight $12, 16, 18, 20, 22, 26$ is. Further, exactly as we saw for $\Delta(\tau)$, for any normalized eigenform f we have

$$\lambda_n = a_f(n).$$

Thus, the coefficients of any normalized eigenform are multiplicative. Thus, the coefficients are determined by the prime power values. The prime powers are actually also determined from the prime values, just in a more complicated way. Specifically, the formulas above imply that we have the following recursion:

$$a_{p^{n+1}}(f) = a_p(f) a_{p^n}(f) - p^{k-1} a_{p^{n-1}}(f) \qquad (n \geq 1).$$

This will have important ramifications when we come back to elliptic curves.

For spaces of dimension at least 2, it will be trickier to find eigenforms. However, we shall see that S_k always has a basis of eigenforms, and that these have "nice" coefficients.