MODULAR FORMS LECTURE 18: HECKE OPERATORS AND NEW MODULAR FORMS FROM OLD

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> There are two facts about the distribution of prime numbers of which I hope to convince you so overwhelmingly that they will be permanently engraved in your hearts. The first is that, despite their simple definition and role as the building blocks of the natural numbers, the prime numbers grow like weeds among the natural numbers, seeming to obey no other law than that of chance, and nobody can predict where the next one will sprout. The second fact is even more astonishing, for it states just the opposite: that the prime numbers exhibit stunning regularity, that there are laws governing their behavior, and that they obey these laws with almost military precision
>
> Don Zagier

1. Hecke operators: First definition

We now return to the multiplicative properties of $\tau(n)$. There is a family of operators, called **Hecke operators**:

$$T_m \colon M_k \to M_k, \quad m = 2, 3, \dots .$$

We write the action of T_m on a form f as either $T_m(f)$ or $f|T_m$. The action of Hecke operators can be defined via the action on Fourier expansions:

$$\left(\sum_n a_n q^n\right) \big| T_m = \sum_n b_n q^n$$

where

$$b_n := \sum_{\substack{r>0 \\ r|(m,n)}} r^{k-1} a_{\frac{mn}{r^2}}.$$

This seems like a strange definition, and its not obvious that it preserves modularity. But we'll see more motivation soon. As a corollary of this formula, we have

$$T_m \colon S_k \to S_k,$$

since $a_0 = 0$ implies that $b_0 = 0$ directly from the function. Thus, as S_{12} has dimension 1, $\Delta(\tau)$ is an **eigenfunction** of each T_m. Suppose that the eigenvalue of T_m is λ_m. Then

$$[q^1](f|T_m) = b_1 = \sum_{\substack{r>0 \\ r|(m,1)=1}} r^{k-1} a_{\frac{m}{r^2}} = a_m = [q^1](\lambda_m f) = \lambda_m a_1.$$

Moreover, coprime Hecke operators **commute**, and satisfy a strict relationship. Specifically

$$T_{mn} = T_m T_n = T_n T_m, \qquad (m,n) = 1.$$

Thus, if f is a **Hecke eigenform** (eigenform of all Hecke operators) then for $(m,n) = 1$ we have $f|T_{mn} = \lambda_{mn} f = f|T_m|T_n = \lambda_m \lambda_n f$ and so $\lambda_{mn} = \lambda_m \lambda_n$. That is, the sequence of eigenvalues λ_n is multiplicative. In particular, taking $f = \Delta$ gives

$$\lambda_{mn} = \tau(mn) = \lambda_m \lambda_n = \tau(m)\tau(n).$$

So assuming the claim on coprime Hecke operators implies Ramanujan's conjecture that $\tau(n)$ is multiplicative, and provides a solid theoretical explanation!

Before moving forward, we'll also note that Eisenstein series are eigenforms. In fact, the coefficients of a modular form are multiplicative if and only if the form is an eigenform. The coefficients $\sigma_{k-1}(n)$ are the summatory function (sum over divisors) of n^k. A basic exercise in elementary number theory shows that since n^k is clearly multiplicative so is its summatory function $\sigma_{k-1}(n)$. Thus, we can break up $M_{12} = \mathbb{C} \cdot E_{12} \oplus \mathbb{C} \cdot \Delta$ which is a basis of eigenforms. We'll see that S_k always has a basis of eigenforms, and hence M_k always will too.

2. New forms from old

Before giving a more theoretical definition of Hecke operators, we study the generally important question of how to build new modular forms from old. We have several methods.

(1) As we've seen, we can add, multiply, divide, and subtract forms.

(2) We have differentiation. We've seen we can use the Serre derivative to stay in the space of holomorhpic modular forms, or leave holomorphic forms using the raising operator. Thanks to Bol's identity, we also saw that D^{k-1} maps from $M^!_{2-k} \to M^!_k$.

There are also the **Rankin-Cohen brackets**. Given $f \in M_k$, $g \in M_\ell$, the n-th bracket is:

$$[f,g]_n = \sum_{\substack{r,s\geq 0\\ r+s=n}} (-1)^{r+s} \binom{k+n-1}{s} \binom{\ell+n-1}{r} D^r f D^s g \in M_{k+\ell+2n}$$

For example,

$$[f,g]_0 = fg,$$

and

$$[f,g]_1 = \ell Dfg - kfDg$$

is a kind of weighted "product rule".

Exercise 1. *Show directly that* $[f,g]_1 \in M_{k+\ell+2}$.

How does the proof of modularity go in general? We saw before what D does when acting on $f(\gamma\tau)$. By a similar calculation and induction, we find

$$D^n f(\gamma\tau) = \sum_{i=0}^{n} \binom{n}{r} \frac{(k+r)_{n-r}}{(2\pi i)^{n-r}} c^{n-r} (c\tau+d)^{k+n+r} D^r f(\tau)$$

Exercise 2. *This is kind of a tricky one, but illustrates what's going on, and is useful to at least read over. The general philosophy of studying sequences of things by sticking them in generating functions and seeing the symmetries of that thing is very powerful. We follow the exposition on Cohen-Kusnetsov series in Zagier's chapter of the 1-2-3 of Modular Forms. Show then that the generating function (for $z \in \mathbb{C}$)*

$$\widetilde{f}(\tau, z) := \sum_{n\geq 0} \frac{D^n f(\tau)}{n!(k)_n} z^n$$

which satisfies (what's nearly a Jacobi form transformation, as we may see later)

$$\widetilde{f}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{(c\tau+d)^2}\right) = (c\tau+d)^k e^{\frac{cz}{(2\pi i)(c\tau+d)}} \widetilde{f}(\tau, z).$$

The inductive formula above may seem familiar; this whole thing can be recast as a generating function of iterated raising operators (see Zagier's chapter). Now, for a modular form g of weight ℓ, check that

$$\widetilde{f}(\tau, -z)\widetilde{g}(\tau, z) = \sum_{n\geq 0} \frac{[f,g]_n z^n}{(k)_n(\ell)_n}$$

gets multiplied by $(c\tau+d)^{k+\ell}$ when $(\tau, z) \mapsto (\gamma\tau, z/(c\tau+d)^2)$. Check that this implies the modularity of $[f,g]_n$.

So differentiation is a natural operation. It turns out that Rankin-Cohen brackets generate all possible operators like this.

(3) The other main way of generating new modular forms is slashing with matrices (possibly with determinant not equal to one). For this, we need the general definition of the slash operator for matrices with general positive determinant:

$$f|\gamma(\tau) := \det(\gamma)^{\frac{k}{2}} (c\tau+d)^{-k} f(\tau).$$

We also need modular forms on subgroups. Usually we stick to congruence subgroups. Let

$$\Gamma(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \ : \ a \equiv d \equiv 1 \pmod N, \ b \equiv c \equiv 0 \pmod N\}$$

be the kernel of reduction mod N (that is, the kernel of the map from $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ given by reducing entries mod N).

Remark. *Note that $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$.*

Definition. A **congruence subgroup** of level N is a subgroup of $\Gamma(1)$ containing $\Gamma(N)$. A modular form on a subgroup Γ of level N (also referred to as a modular form of level N) is a function which is holomorphic on $\mathbb{H}$, such that $f|_k\gamma = f$ for all $\gamma \in \Gamma$, and which is "holomorphic at the cusps". Explicitly, this means that for all $\gamma' \in \Gamma(1)$, there is an expansion of the shape

$$f|\gamma'(\tau) = \sum_{n\geq 0} a_n q^{\frac{n}{N}}.$$

It is a cusp form if each of these expansions has $a_0 = 0$. The space of modular forms of weight k on Γ is denoted $M_k(\Gamma)$, and the space of cusp forms as $S_k(\Gamma)$.

Remark. *We have $\Gamma(N') \subseteq \Gamma(N)$ if N' is a multiple of N. So a level N group is also a level N' group. Thus, a modular form of level N is also a modular form of level aN' for any $a \geq 1$.*

What about the *minimal* possible level of a form? We will discuss this later.

Modular forms on $\Gamma(N)$ are more difficult to study than on $\Gamma(1)$. The theory is simpler if we stick to a nice congruence subgroups of level N like

$$\Gamma_0(N) := \left\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \pmod N\right\}.$$

Since these are so common to study, we also write $M_k(\Gamma_0(N)) = M_k(N)$. The next nicest class of congruence subgroups are those of the form

$$\Gamma_1(N) := \left\{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \pmod N\right\}.$$

Modular forms on these groups are nearly as nice as those for $\Gamma_0(N)$. The reason is that there is a decomposition

$$M_k(\Gamma_1(N)) = \oplus_\chi M_k(\Gamma_0(N), \chi)$$

into spaces of modular forms with **Nebentypus**, which just means that the automorphy factor is twisted by a Dirichlet character χ.

Why is $\Gamma_0(N)$ so nice? We have $T \in \Gamma_0(N)$, but its not in $\Gamma(N)$. However, T^N is, which is why in the above definition of holomorphicity at the cusps, the expansions were in terms of $q^{\frac{1}{N}}$; we only have invariance under $\tau \mapsto \tau + N$ in general.

Why are congruence subgroups so special?

Conjecture. *Only congruence subgroup modular forms can have integral (or bounded denominator in $\mathbb{Q}$) coefficients.*

So if you want something that counts something like partitions, you could only possibly find congruence subgroups. However, there are deep applications of more general modular forms to geometry, though they're less well-understood. An illustration of how much these groups encode is given by the following.

Theorem (Belyi)**.** *Every smooth irreducible projective plane curve over $\mathbb{Q}$ is isomorphic to $\Gamma\backslash\mathbb{H}$ for some subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$.*

As the level grows, the groups and their modular forms get more complicated. The **cusps** for a group are the equivalence classes of $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{i\infty\}$ under the action of Γ. Modular forms have to be bounded at all of these cusps, and cusp forms have to vanish at all of these cusps. For $\Gamma(1)$, there is only one cusp, as it acts transitively on $\mathbb{P}^1(\mathbb{Q})$ (you can get from any point to any other point in the set with a Möbius transformation in $\Gamma(1)$).

If $f \in M_k(\Gamma)$, then slashing with $\gamma \in \Gamma$ doesn't do anything. Slashing with more general things doesn't return the same function, but instead returns a new modular form.

Proposition. *Let $\Gamma' \leq \Gamma(1)$ be a congruence subgroup. For $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ (a rational matrix with positive determinant, which thus acts on $\mathbb{H}$), set*

$$\Gamma'' := \alpha^{-1}\Gamma'\alpha \cap \mathrm{SL}_2(\mathbb{Z}).$$

Then the following are true.
(a) *Γ'' is a congruence subgroup.*
(b) *If $f \in M_k(\Gamma')$, then $f|\alpha \in M_k(\Gamma'')$. Moreover, if $f \in S_k(\Gamma')$, then $f|\alpha \in S_k(\Gamma'')$.*

Proof. Multiplying α by a constant doesn't change the slash action, so without loss of generality, let $\alpha \in \mathrm{GL}_2^+(\mathbb{Z})$. We claim that if $D = \det(\alpha)$, then

$$\Gamma' \supseteq \Gamma(N) \implies \alpha^{-1}\Gamma'\alpha \supseteq \Gamma(ND).$$

Indeed, if $\gamma \in \Gamma(ND)$, then $\gamma = 1 + ND\beta$ for $\beta \in \Gamma(1)$. We need to show that $\gamma \in \alpha^{-1}\Gamma'\alpha$, i.e., that Γ' contains

$$\alpha\gamma\alpha^{-1} = \alpha(1 + ND\beta)\alpha^{-1} = 1 + ND\alpha\beta\alpha^{-1}.$$

Now $\alpha' = D\alpha^{-1}$ is an integer matrix, and $\det(\alpha\gamma\alpha^{-1}) = \det(\gamma) = 1$. Moreover,

$$\alpha\gamma\alpha^{-1} = 1 + N\alpha\beta\alpha' \implies \alpha\gamma\alpha^{-1} \in \Gamma(N) \subseteq \Gamma',$$

proving (1).

For (2), the modularity transformations are clear as

$$(f|\alpha)|(\alpha^{-1}\gamma'\alpha) = f|(\alpha\alpha^{-1})|\gamma'|\alpha = (f|\gamma')|\alpha = f|\alpha.$$

We have to show that if f is holomorphic at the cusps, then so is $f|\alpha$, and similarly, that if f vanishes at the cusps, then so does $f|\alpha$.

Exercise 3. *Show that there exists $\gamma_0 \in \Gamma(1)$ so that $\gamma^{-1}\alpha = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with $a, d \in \mathbb{N}$.*

Using this exercise, and the assumption that f has an appropriate Fourier expansion at the cusps as discussed in the definition of holomorphicity at cusps,

$$f|\alpha = (f|\gamma_0)|\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = (ad)^{\frac{k}{2}} d^{-k} \sum_{n\geq 0} a_n e\left(\frac{n}{N}\frac{a\tau+b}{d}\right)$$

$$= \left(\frac{a}{d}\right)^{\frac{k}{2}} \sum_{n\geq n_0} a_n e\left(\frac{bn}{dN}\right) e\left(\frac{an\tau}{Nd}\right) = \left(\frac{a}{d}\right)^{\frac{k}{2}} \sum_{n\geq n_0} a_n e\left(\frac{bn}{dN}\right) q^{\frac{an}{Nd}}$$

has the right shape of a Fourier expansion. This shape also implies that cuspidality is also preserved. □

Important special cases

As examples, we have the V**-operator**

$$f|V_m = m^{-\frac{k}{2}} f\Big| \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} = f(m\tau)$$

and the U**-operator**

$$f|U_m = m^{\frac{k}{2}-1} \sum_{j=1}^{m} f\Big| \begin{pmatrix} 1 & j \\ 0 & m \end{pmatrix}.$$

On Fourier expansions, we have.

$$\sum_n a_n q^n | V_m = \sum_n a_n q^{mn}, \qquad \sum_n a_n q^n | V_m = \sum_n a_{mn} q^n.$$

The expansion is clear for V_m, and for U_m, we find

$$f|\left(\begin{smallmatrix} 1 & j \\ 0 & m \end{smallmatrix}\right)(\tau) = m^{\frac{k}{2}} m^{-k} f\left(\frac{\tau+j}{m}\right) = m^{-\frac{k}{2}} \sum_n a_n e\left(\left(\frac{\tau+j}{m}\right) n\right) = m^{-\frac{k}{2}} \sum_n a_n e\left(\frac{jn}{m}\right) q^{\frac{n}{m}}.$$

Thus,

$$f|U_m = m^{\frac{k}{2}-1} \sum_{j=1}^{m} m^{-\frac{k}{2}} \sum_n a_n e(jn/m) q^{n/m} = m^{-1} \sum_n a_n q^{n/m} \sum_{j=1}^{m} e(jn/m).$$

This exponential sum is 0 if $m \nmid n$, and m if $m|n$. Thus, this becomes

$$\sum_{m|n} a_n q^{n/m} = \sum_n a_{mn} q^n.$$

On spaces, we have from the above that U_m, V_m map $M_k(N)$ to $M_k(mN)$. So the level goes up, but not so much. But sometimes you're lucky!

Exercise 4. *Let $f \in M_k(4)$ with k even. Show that $f|U_2 \in M_k(2)$. Further show if $a_n = 0$ with $n \equiv 2 \pmod 4$, then $f|U_4 \in M_k(1) = M_k$.*

This little exercise has important uses; it naturally occurs when studying Rankin-Cohen brackets of two half-integral weight modular forms, and its useful in the important Eichler-Selberg trace formula (see Zagier's "Intro to modular forms" in From Number Theory to Physics).