MODULAR FORMS LECTURE 17: PARTITIONS

LARRY ROLEN, VANDERBILT UNIVERSITY, FALL 2020

> If we compared the Bernoullis to the Bach family, then Leonhard Euler is unquestionably the Mozart of mathematics.
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> Eli Maor, e: The Story of a Number (1994)

1. Definitions and generating function

There are many applications of the Dedekind eta function. One that stands out is the theory of **integer partitions**. Following Euler, consider the infinite product $(q)_\infty^{-1}$ and expand out terms using geometric series

$$\frac{1}{(q)_\infty} = \prod_{n\geq 1} \frac{1}{1-q^n} = (1+q+q^2+\ldots)\cdot(1+q^2+q^4+\ldots)(1+q^3+q^6+\ldots)\cdot\ldots.$$

The coefficient of q^n in this expansion is the number of ways to write n as a sum of smaller natural numbers, which we call the number of **partitions** of n. For example

$$10 = 5+2+2+1,$$

and this corresponds to the contribution to the coefficient q^{10} coming from choosing 1 one (q^1 in the first product), 2 two's ($(q^2)^2$ in the second product), 0 three's (q^0 in the third product), 0 four's (q^0 in the fourth product), one 5, and 0 of every other number. The choices are displayed in red:

$$(1+\color{red}{q}+q^2+\ldots)\cdot(1+q^2+\color{red}{q^4}+\ldots)(\color{red}{1}+q^3+q^6+\ldots)\cdot(\color{red}{1}+q^4+q^8+\ldots)\cdot(1+\color{red}{q^5}+q^{10}+\ldots)\cdot(\color{red}{1}+\ldots)\cdot(\color{red}{1}+\ldots)\ldots.$$

Let $p(n)$ be the number of partitions of n, and $P(q) := \sum_{n\geq 0} p(n)q^n$. Then

$$P(q) = \frac{1}{(q)_\infty} = \frac{q^{\frac{1}{24}}}{\eta(\tau)},$$

essentially a weight $-1/2$ weakly holomorphic modular form.

2. Asymptotics

Modularity of the generating function for $p(n)$ implies *many* things. For example, how fast $p(n)$ grows, which was a really hard problem at the time.

Theorem (Hardy-Ramanujan). *We have*

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}.$$

Here, the asymptotic $\sim$ means that the quotient of the two sides tends to 1 as $n \to \infty$.

Modularity is completely key to the proof. Essentially, you need the behavior of $P(q)$ as $q \to 1$ radially inside the unit disk. Hardy and Ramanujan invented the Circle Method to solve this, which computes the coefficients via integrals that get very close to the singularities of $1/(q)_\infty$ at the roots of unity in a very clever way. That method is a fundamental one in number theory today, and applies outside of modular forms too; Helfgott recently used it to prove the weak Goldbach Conjecture that odd numbers greater than 5 are sums of 3 primes.

But we can give a more explicit check in this case, largely due to the fact that $p(n)$ is easier to study since its monotonic. This will still illustrate the general philosophy.

Specifically, we have

$$p(n) \leq p(n+1)$$

as there is an obvious injection

$$\{\text{ partitions of n}\} \hookrightarrow \{\text{ partitions of n+1}\}$$

given by

$$n = n_1 + \ldots + n_k \mapsto n+1 = n_1 + \ldots + n_k + 1.$$

I always think a theory is justified if you can say you can prove something using the theory that you could have understood the statement of externally to the subject, but that direct approaches without the new theory wouldn't suffice. In the case of $p(n)$, how would you estimate the growth rate without modular forms? A nice example is that you can show the exponential growth rate as a lower bound, just with the wrong constant in the exponential. Specifically, there is a surjection onto the power set:

$$\{\text{partitions of } n\} \twoheadrightarrow 2^{\{1,2,\ldots,\lfloor\sqrt{n}\rfloor\}}$$

given by sending $n = n_1 + \ldots + n_k$ to the set of those numbers up to $\sqrt{n}$ which appear among the n_i. This is actually surjective as

$$1 + 2 + 3 + \ldots + \lfloor\sqrt{n}\rfloor \leq \frac{(\sqrt{n})^2 + \sqrt{n}}{2} = \frac{n + \sqrt{n}}{2} < n.$$

So given a subset, you can add up all the numbers in it and then add one extra number to it to make the numbers all add to n, yielding a preimage.

Returning to modular forms, Ingham developed the following Tabuerian Theorem (these have a long history in analysis, for example in the proof of the Prime Number Theorem) to study situations like ours.

Theorem (Tabuerian Theorem of Ingham)**.** *Let $f(q) = q^\alpha \sum_{n\geq 0} a(n)q^n$ be a q series holomorphic on $\mathbb{H}$. Suppose that*

1. *$a(n)$ is non-negative and monotonic*
2. *There are numbers $N > 0, c, d \in \mathbb{R}$ such that the following asymptotic holds:*

$$f(e^{-\varepsilon}) \sim c\varepsilon^d e^{\frac{N}{\varepsilon}}, \qquad \text{as } \varepsilon \searrow 0.$$

This limit is taking along a vertical line, namely, along the imaginary axis.

Then we have the following asymptotic:

$$a(n) \sim \frac{c}{2\sqrt{\pi}} \frac{N^{\frac{d}{2}+\frac{1}{4}}}{n^{\frac{d}{2}+\frac{3}{4}}} e^{2\sqrt{Nn}}.$$

In the case of partitions, letting $\widetilde{P} := 1/\eta$ and specializing to $\tau = it \in i\mathbb{R}$, we have

$$\eta(-1/\tau) = \sqrt{\frac{\tau}{i}}\eta(\tau) \implies \eta\left(\frac{i}{t}\right) = \sqrt{t}\eta(it) \implies \widetilde{P}(i/t) = t^{-\frac{1}{2}}\widetilde{P}(it).$$

Thus,
$$P(i/t) = q^{\frac{1}{24}}\Big|_{\tau=i/t}\widetilde{P}(i/t) = e^{-\frac{2\pi}{24t}}t^{-\frac{1}{2}}\widetilde{P}(it).$$
Now
$$\widetilde{P}(it) = e^{\frac{-2\pi t}{24}}(1+\dots) \sim_{t\searrow 0} e^{-\pi t/24} \sim 1 \implies P(i/t) \sim e^{-\frac{\pi}{12t}}t^{-\frac{1}{2}}.$$
Plugging into Ingham's theorem gives the Hardy-Ramanujan asymptotic.

Other analytic results include properties like the fact due to Nicolas and DeSalvo-Pak that $p(n)$ is eventually log-concave:
$$p(n)^2 - p(n-1)p(n+1) \geq 0, \qquad n > 25.$$
Many other interesting inequalities for $p(n)$ also hold. This is much easier to show if you don't worry about making it effective.

Exercise 1. *Use the Hardy-Ramanujan asymptotic to show that $p(n)$ is eventually log-concave (without deducing the explicit constant 25).*

3. Congruences

Ramanujan conjectured the following remarkable congruences
$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$
We may prove these later once we know more stuff. These were shortly proven by Hardy based on Ramanujan's notes, but you may ask what their deeper meaning is. This has a beautiful story. The famous physicist Freeman Dyson had a brilliant idea. He defined the **rank** of a partition as the largest part minus the number of parts. Amazingly, he observed (later proven by Atkin and Swinnerton-Dyer) that the rank mod 5 splits up partitions of $5n+4$ into 5 equally sized sets, and similarly modulo 7!

Example 1. We have $p(4) = 5$. The partitions of 5 are $4, 3+1, 2+2, 2+1+1, 1+1+1+1$. The ranks, respectively are $4-1=3, 3-2=1, 2-2=0, 2-3=-1, 4-1=3$. Mod 5, the ranks are, respectively $3, 1, 0, 4, 2$. Thus, each congruence class is represented exactly once.

Example 2. We have $p(5) = 7$. The partitions of 5 are $5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1$. These have ranks $4, 2, 1, 0, -1, -2, 3$. Modulo 7, these numbers are $4, 2, 1, 0, 6, 5, 3$, and each congruence class is represented exactly once.

Dyson's rank doesn't explain the congruences mod 11, but there is another invariant, which Dyson reserved the name "crank" for, later found by Andrews and Garvan.

What about congruences mod other primes? Well, it turns out, as recently shown by Radu, that there are **no linear congruences** modulo 2 or 3. This was hard to prove and a long-standing conjecture. We expect that $p(n)$ is equidistribued mod 2 and 3, but we are very far from proving this (we are even very far from showing that $p(n)$ is even or odd more than 0% of the time). The best we can show is that the number of partition values up to x which are even or odd is at least $\gg X^{\frac{1}{2}+\varepsilon}$.

It also turns out that there are no *simple* congruences mod other primes for $p(n)$. That is, Ahlgren and Boylan showed that if $p(\ell n + a) \equiv 0 \pmod{\ell}$ for a prime ℓ, then $\ell \in \{5, 7, 11\}$.

4. Chowla-Selberg Formula

There are also explicit results for evaluating the eta function at special places. For instance, we have

$$\eta(i) = \frac{1}{2\pi^{\frac{3}{4}}}\Gamma(1/4).$$

Using the theory of **complex multiplication** we'll discuss later, and using zeta functions, one can show the following (this would make a possible final project for the class):

$$\prod_{j=1}^{h} a_j^{-6}\Delta(\tau_j) = (2\pi|d|)^{-6h}\left[\prod_{m=1}^{|d|}\Gamma\left(\frac{m}{|d|}\right)^{\frac{d}{m}}\right]^{3w}$$

where the product is over a complete set of **reduced** binary quadratic forms (a_i, b_i, c_i) of discriminant $d < 0$ of class number h, where $\tau_j := \frac{b_j+\sqrt{d}}{2a_j}$. Further, w is half the number of roots of unity in $\mathbb{Q}(\sqrt{d})$ (its almost always 1). We'll talk much more about these terms later. For now, I just wanted to give an idea of the shape of the formula.

5. Eta as theta

Euler also proved his **pentagonal number theorem**:

$$(q)_\infty = \sum_{k\in\mathbb{Z}}(-1)^k q^{\frac{k(3k+1)}{2}}.$$

Its always best to complete the square, and when we do that, we find with $\chi_{12} := \left(\frac{12}{\cdot}\right)$:

$$(q)_\infty = \sum_{n\geq 1}\chi_{12}(n) q^{\frac{n^2-1}{24}}.$$

This represents $\eta(\tau)$ as a **theta function**, which we will soon have a lot more to say about. Essentially, this are the other way we build up modular forms. Eisenstein series are group averages, and theta functions are modular thanks to a trick known as Poisson summation. For now, you can think that a theta function is something like a sum over a lattice of a "character" like χ_{12} times q raised to a quadratic power.

The Pentagonal Number Theorem tells us something nice about partitions:

$$P(q)\sum_{k\in\mathbb{Z}}(-1)^k q^{\frac{k(3k+1}{2}} = 1.$$

Thus, we get a *recursion* for $p(n)$:

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + \ldots.$$

This is not a fast way using current knowledge to compute a large value of $p(n)$, but its still a very efficient way, if not the best way, to compute *tables* of all values up to n. MacMahon famously used this to compute

$$p(200) = 3972999029388.$$

Hardy and Ramanujan used this to check their asymptotic.