MODULAR FORMS LECTURE 16: THE DEDEKIND ETA FUNCTION

LARRY ROLEN, VANDERBILT UNIVERSITY, FALL 2020

> Je le vois, mais je ne le crois pas! (I see it, but I don't believe it!)
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> Cantor to Dedekind on his proof that an interval has the same cardinality as a square

We have seen that $\Delta = q(q)_\infty^{24}$ is a cusp form of weight 12. What if we didn't include the 24? Then we would have the **Dedekind eta function**:

$$\eta(\tau) := q^{\frac{1}{24}}(q)_\infty.$$

If this were modular, we'd expect the weight to be 1/2. But, no modular forms of weight 1/2 exist under our definition! Here's why.

Let $\sqrt{\cdot}$ be a principal branch of the square root. In order for $f(\gamma\tau) = j(\gamma;\tau)f(\tau)$ to make sense, we saw that j must be an *automorphy factor*:

$$j(\gamma_1\gamma_2;\tau) = j(\gamma_1;\gamma_2\tau)j(\gamma_2;\tau).$$

The "naive guess" in half integral weight is $j(\gamma;\tau) = (c\tau+d)^{\frac{k}{2}}$, where k is odd. This is **not** an automorphy factor. Consider the matrices

$$\gamma_1 := \begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix} \qquad \gamma_2 := \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}.$$

Then the automorphy factor relation implies that

$$\sqrt{3\tau-2} = \sqrt{\frac{-3\tau}{1-3\tau}-2}\cdot\sqrt{1-3\tau}.$$

The square of this relation holds, so the k-th power holds up to a sign. But its the *wrong sign*. The right hand side is a product of square roots of things in the lower half plane:

$$\text{RHS} \in \sqrt{\mathbb{H}^-}\sqrt{\mathbb{H}^-} \implies \text{RHS} \in \text{ 4th quadrant} \cdot \text{ 4th quadrant}.$$

But the left hand side is in the first quadrant, as $3\tau - 2 \in \mathbb{H}$. So in fact the relation is off by a factor of -1.

Remark. *Once we talk about congruence subgroups of* $\mathrm{SL}_2(\mathbb{Z})$*, one can check by a similar calculation that the condition fails for those too.*

But there is a small correction with roots of unity to fix things. This is

$$j(\gamma;\tau) := \left(\frac{c}{d}\right)\varepsilon_d^{-1}\sqrt{c\tau+d}.$$

Here, $\left(\frac{c}{d}\right)$ is a **Kronecker character** (this is a basic function from elementary number theory that only takes the values $0, \pm1$, 0 exactly when $(c,d) \neq 1$) and

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

This is an automorphy factor for the **level** 4 **congruence subgroup** $\Gamma_0(4)$, where

$$\Gamma_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \ : \ N|c \right\}.$$

Half-integral weight modular forms using this automrphy factor are only defined on levels divisible by 4. This allows us to make the following.

Definition. A modular form of weight $k/2$ with k odd on $\Gamma_0(4N)$ is a function $f\colon \mathbb{H} \to \mathbb{C}$ that is is holomorphic on $\mathbb{H}$ and at infinity and such that for all $\gamma \in \Gamma_0(4N)$ we have

$$f(\gamma\tau) = j(\gamma;\tau)^k f(\tau).$$

It turns out that $\eta(\tau)$ is a modular form of weight $1/2$ on $\mathrm{SL}_2(\mathbb{Z})$, but we have to twist the transformations by roots of unity to get what's known as a **multiplier system**. Non-integral powers of q in the Fourier expansion translate to non-translation invariance. But that's an easy fix. The $q^{\frac{1}{24}}$ just gives us

$$\eta(\tau+1) = e(1/24)\eta(\tau),$$

where $e(x) := e^{2\pi i x}$. It turns out that

$$\eta(-1/\tau) = \sqrt{\frac{\tau}{i}}\eta(\tau).$$

To check this, simply take the log derivative as we did for Δ to find

$$\frac{\eta'}{\eta} = \frac{i}{4\pi} G_2.$$

Since $G_2(-1/\tau) = \tau^2 G_2(\tau) - 2\pi i \tau$, with $F := \eta'/\eta$, we have

$$f(-1/\tau)\cdot\frac{1}{\tau^2} - f(\tau) - \frac{1}{2\tau} = 0.$$

For $v > 0$, set

$$g(v) := \frac{\eta(i/v)}{\eta(iv)\sqrt{v}}.$$

Then

$$\frac{g'(v)}{g(v)} = f(i/v) - \frac{i}{v^2}\cdot(-i) - f(iv) - \frac{1}{2v} = 0 \implies \eta(i/v) = C\sqrt{v}\eta(iv)$$

for some C. For $v = 1$, we have $c = 1$. Thus, the inversion property holds on the imaginary axis intersect $\mathbb{H}$. By the **Identity Theorem** from complex analysis, this holds everywhere on $\mathbb{H}$.

Finally, you can combine the relations at S and T to get the full multiplier system for $\eta(\tau)$. We won't prove this, but it involves **Dedekind sums**. Define the **sawtooth function**

$$((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & x \in \mathbb{R}\setminus\mathbb{Z}, \\ 0 & x \in \mathbb{Z}. \end{cases}$$

Then the Dedekind sum is

$$s(a,b) := \sum_{n \pmod b} \left(\left(\frac{n}{b}\right)\right)\left(\left(\frac{an}{b}\right)\right).$$

Then the full transformation behavior for $\eta(\tau)$ on $\mathrm{SL}_2(\mathbb{Z})$ is

$$\eta(\gamma\tau) = \varepsilon(\gamma)(c\tau + d)^{\frac{1}{2}}\eta(\tau),$$

where $\varepsilon(\gamma)$ is the multiplier system

$$\varepsilon(\gamma) = \begin{cases} e(b/24) & \text{if } c = 0, d = 1, \\ e\left(\frac{a+d}{24c} - \frac{s(d,c)}{2} - \frac{1}{8}\right) & \text{if } c > 0. \end{cases}$$