MODULAR FORMS LECTURE 15: THE RAMANUJAN τ FUNCTION AND GROWTH RATES OF FOURIER COEFFICIENTS

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> I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavorable omen. "No", he replied, "it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."
>
> Hardy, about a visit to Ramanujan

1. The product formula for Δ

We begin with an alternate representation of the modular discriminant function. For now, we'll take this as a new definition, but we'll soon see that its the same as before. Set

$$\Delta(\tau) := q\prod_{n\geq 1}(1-q^n)^{24} = q\cdot (q)_\infty^{24},$$

where

$$(a;q)_n := (a)_n = \prod_{j=0}^{n-1}(1-aq^j)$$

is the q-**Pochhammer symbol**. In particular, note that Δ has no zeros on $\mathbb{H}$, since its an infinite product and the n-th term in the product is 0 if and only if $q^n = 1$. Thus, the zeros are precisely the set of τ where $q = 0$ or q is a root of unity; namely, $\tau \in \mathbb{Q}\cup\infty =: \mathbb{P}^1_{\mathbb{Q}}$.

Theorem. *We have that $\Delta \in S_{12}$.*

Corollary. *Since both are weight 12 cusp forms with constant term 1, we have*

$$\frac{E_4^3 - E_6^2}{1728} = q(q)_\infty^{24}.$$

Proof. Cuspidality is clear as $q|\Delta$. Consider the log derivative, which turns the product into a sum, and is sensible since Δ has no zeros on $\mathbb{H}$:

$$\frac{\Delta'(\tau)}{\Delta(\tau)} = D\log\left(q\prod_{n\geq 1}(1-q^n)^{24}\right) = 1 - 24\sum_{n\geq 1}\frac{nq^n}{1-q^n}.$$

This is an example of a **Lambert series**. Note that

$$\frac{q^n}{1-q^n} = \sum_{k\geq 1} q^{nk}.$$

More generally, we have

$$\sum_{n\geq 1}\frac{a_nq^n}{1-q^n} = \sum_{n\geq 1}a_n\sum_{k\geq 1}q^{nk} = \sum_{m\geq 1}\sum_{n|m}a_nq^m$$

by rearranging summation. In particular,

$$\frac{\Delta'}{\Delta} = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n = E_2(\tau).$$

We can deduce the transformation law for Δ using our transformation for E_2:

$$D\log\left(\frac{\Delta(\gamma\tau)}{(c\tau+d)^{12}\Delta(\tau)}\right) = \frac{1}{(c\tau+d)^2}E_2(\gamma\tau) - \frac{12}{2\pi i}\cdot\frac{c}{c\tau+d} - E_2 = 0.$$

That is,

$$\Delta|_{12}\gamma = C(\gamma)\Delta$$

for all $\gamma\in\Gamma$ and $C(\gamma)$ a complex number. It suffices to show that each $C(\gamma)=1$. Since $\Delta\mapsto\Delta_{12}\gamma$ is a group action, its enough to show $C(T)=C(S)=1$. Now $C(T)=1$ as Δ is a function of q *by definition* so is periodic. Further,

$$\Delta(-1/\tau) = C(s)\tau^{12}\Delta(\tau)$$

and plugging in at $\tau = i$ gives

$$\Delta(i) = C(S)\Delta(i).$$

Since $\Delta(i)\neq 0$, we have $C(S)=1$. □

2. The Ramanujan τ function

The coefficients of Δ form the **Ramanujan τ function**:

$$\Delta(\tau) =: \sum_{n\geq 1}\tau(n)q^n.$$

The first few coefficients are:

$$\Delta(\tau) = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6.$$

Note in particular that

$$\tau(2)\tau(3) = (-24)\cdot 252 = -6048 = \tau(2\cdot 3) = \tau(6).$$

This is not a coincidence! Ramanujan famously conjectured the following.

Conjecture (Ramanujan)**.** *The function $\tau(n)$ is* **multiplicative**, *that is, when $(m,n)=1$, we have*

$$\tau(mn) = \tau(m)\tau(n).$$

We will prove this via the theory of **Hecke operators**.

There are also deep analytic properties.

Conjecture 1 (Ramanujan)**.** *We have that*

$$\tau(n) = O\left(n^{\frac{11}{2}+\varepsilon}\right).$$

Even better, for prime p, we have

$$|\tau(p)| \leq 2p^{\frac{11}{2}}.$$

Remark. *We'll see later that the behavior at primes determines all values of τ.*

We will **not** be able to prove that in this class, as it requires Deligne's proof of the Weil conjectures (he won the Fields medal for this). Though we will discuss the Weil Conjectures later when we return to elliptic curves. It should be noted that this conjecture was **extremely** influential and is the first of a general type of conjecture in analytic number theory.

3. Growth of modular form Fourier coefficients

We can, however, get close to Ramanujan's conjectured growth, and prove the estimate

$$\tau(n) = O\left(n^{\frac{12}{2}}\right) = O(n^6).$$

More generally,

Theorem. *If $f \in S_k$, then the Fourier coefficients $a_f(n)$ satisfy*

$$|a_f(n)| \leq Cn^{\frac{k}{2}}$$

for some C.

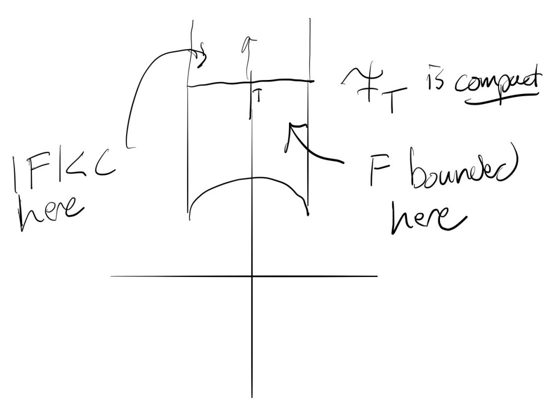
Proof. First I'll leave you with an imperative (but not that difficult) fact to check. We'll need something similar later to define inner products.

Important Exercise 1. *The function $F(\tau) := v^{\frac{k}{2}}|f(\tau)|$ is Γ-invariant.*

Now since $f \in S_k$, we have $f = O(q)$. As

$$q\Big|_{\tau=iv} = e^{-2\pi v},$$

f decays exponentially as $v \to \infty$. On the fundamental domain, F is bounded. To see this, note that for any $C > 0$ and for v large enough, say above height T, then $|F| < C$, but the truncated fundamental domain $\mathcal{F}_T$ at height T is compact so F is bounded there too.



By Γ-invariance, F is bounded on $\mathbb{H}$. Let's say

$$F(\tau) = v^{\frac{k}{2}}|f(\tau)| \leq C' \implies |f(\tau)| \leq C'v^{-\frac{k}{2}}.$$

Now we can recover the Fourier coefficients with the following fundamental formula

$$a_f(n) = e^{2\pi nv} \int_0^1 f(u+iv)e^{-2\pi inu}du \qquad \text{(for any } v > 0\text{)}.$$

Thus,

$$|a_f(n)| \leq C'v^{-\frac{k}{2}}e^{2\pi nv}.$$

We can even choose v optimally by setting $v := \frac{k}{4\pi n}$ (but getting this optimally isn't required; the key point is that we choose $v \approx n^{-1}$ so that the v power is the $n^{\frac{k}{2}}$ in the theorem and the exponential is a constant), which then gives the statement of the theorem with

$$C = C'\left(\frac{4\pi e}{k}\right)^{\frac{k}{2}}.$$

□

Corollary. *If $f \in M_k$ then*

$$|a_f(n)| \ll n^{k-1+\varepsilon}.$$

Remark. *The Vinagradov notation $\ll \cdot$ just means $O(\cdot)$ but its often handy (especially to denote dependence on certain parameters and string it in chains).*

Remark. *The general version of Ramanujan's conjectures then says that the growth of cusp form coefficients is like the square root of the growth of non-cusp forms.*

Proof. Since $M_k = \mathbb{C} \cdot E_k \oplus S_k$, and we know how to estimate the growth of coefficients in S_k, we look at E_k. For $n > 1$, we have

$$[q^n]E_k \doteq \sigma_{k-1}(n) \approx n^{k-1}.$$

Why is this true? Well, we clearly have the bounds (the first since $1, n$ are divisors of n, the second because the set of divisors of n is a subset of $\{1, 2, \ldots, n\}$; for the final estimate compare this to an integral)

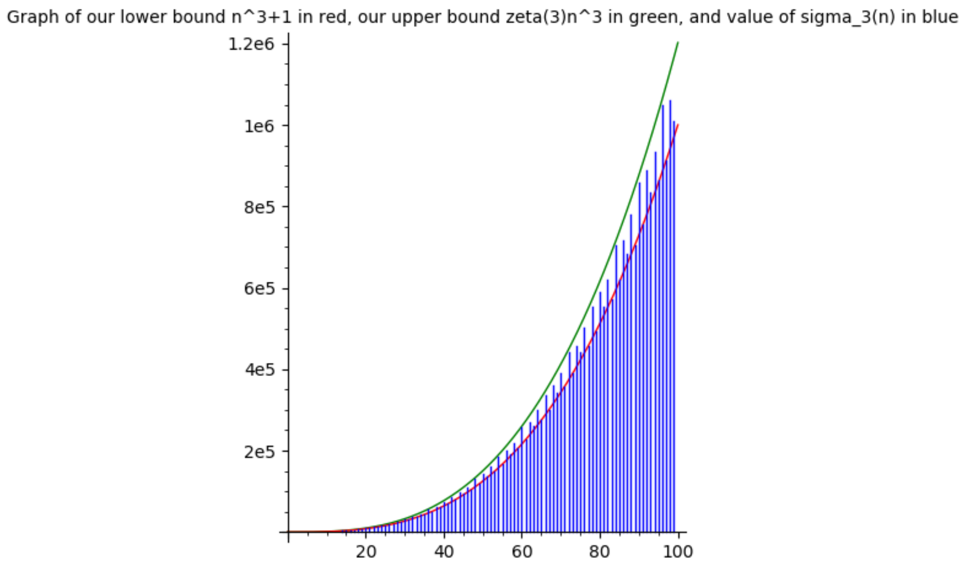
$$1 + n^{k-1} \leq \sigma_{k-1} \leq \sum_{j=1}^{n} j^{k-1} = O(j^k).$$

But we can do better on the upper bound. Every divisor of n has a **complementary divisor** n/d, and as d ranges over all divisors, so does n/d. Thus,

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1} = \sum_{d|n} \left(\frac{n}{d}\right)^{k-1} = n^{k-1}\sum_{d|n} d^{-(k-1)} \leq n^{k-1}\sum_{d\geq 1} d^{-(k-1)} = \zeta(k-1)n^{k-1} = O(n^{k-1}).$$

This is much larger than the maximum growth of cusp form coefficients, so it gives an asymptotic upper bound on the growth of any modular form. □

Thus, for non-cusp forms, the growth is dominated by the growth coming from the Eisenstein series, which doesn't oscillate too much but goes back and forth between multiples of n^{k-1}. Here is a plot for the first coefficients of G_4:



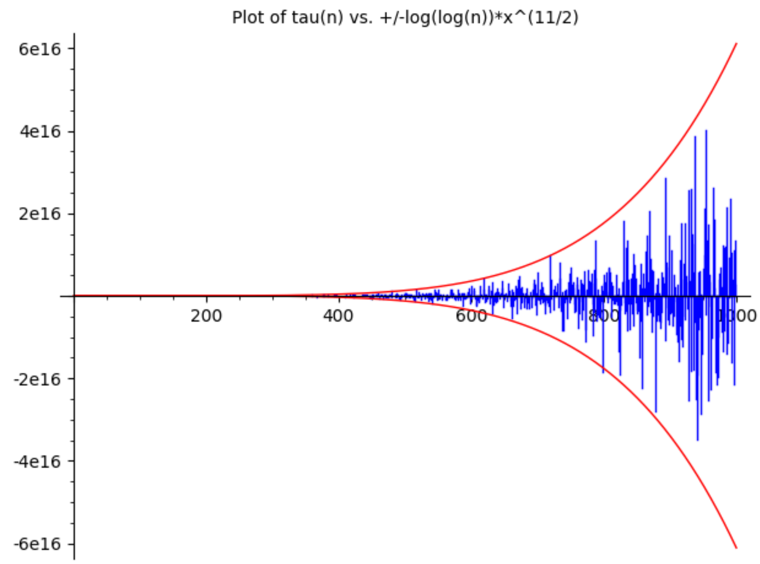
What about **explicit, tight** upper bounds for the coefficients of non-cusp forms, rather than big-Oh estimates or estimates that include a lot of extra terms like we did by bounding it with a ζ function ? Well to give you an idea of the difficulty, consider the analogous problem for E_2. There you'd want a good upper bound for $\sigma_1(n)$. But we have the following.

Theorem (Lagarias)**.** *The Riemann Hypothesis is equivalent to the statement that*

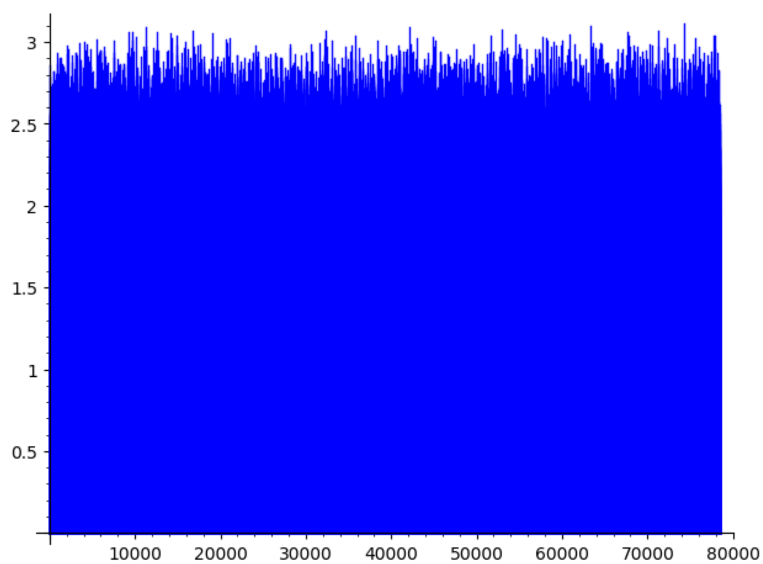
$$\sigma_1(n) \leq H_n + e^{H_n} \log(H_n)$$

for all $n \geq 1$ *with equality only for* $n = 1$*, where* H_n *is the* n*-th harmonic number* $H_n := \sum_{j=1}^{n} \frac{1}{j}$.

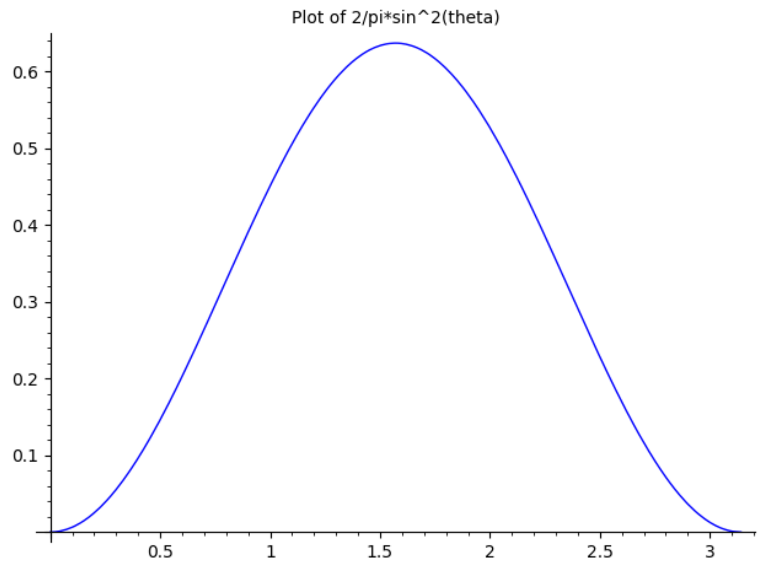
How do the coefficients of cusp forms really grow? Let's look at Δ. I've plotted an approximation of what you get from Ramanujan's conjecture, which shows huge amounts of oscillation in the coefficients:



How do these coefficients actually vary in the range $[-n^{11/2+\varepsilon}, n^{11/2+\varepsilon}]$? Well, let's consider the prime values which we'll see determine everything. Then we know that $\tau(p) \in [-2p^{11/2}, 2p^{11/2}]$. So $\tau(p)/(2p^{11/2}) \in [-1, 1]$. Thus, $\tau(p)/(2p^{11/2}) = \cos(\vartheta_p)$ for some angle $\vartheta_p \in [0, 2\pi)$. Let's plot these angles for the primes up to one million (the n-th coordinate is the n-th prime):



The values are certainly between 0 and π, but they don't seem equally distributed. I tried to have my computer make histograms, but it was too slow to get compute enough for a good picture. But, if you do draw this picture, you find that the angles are distributed like $\frac{2}{\pi}\sin^2(\vartheta)$:



Then we have the following conjecture, whose proof was a very major recent theorem.

Theorem (Sato-Tate Conjecture; Now a theorem of Barnett-Lamb, Geraghty, Harris, and Taylor)**.** *If $f(\tau)$ is a certain type of cusp form such as $\Delta(\tau)$, then the angles ϑ_p defined at primes p by*

$$\frac{a_f(p)}{2p^{\frac{k-1}{2}}} = \cos(\vartheta_p)$$

satisfy the following distribution. If $[a,b] \subseteq [0,\pi]$ is any interval, then

$$\lim_{x\to\infty} \frac{\#\left\{p \le x \;\middle|\vartheta_p \in I\right\}}{\#\{p \le x \mid p \text{ is prime}\}} = \int_a^b \frac{2\sin^2\vartheta d\vartheta}{\pi}.$$

4. A famous open problem

There is a very famous conjecture in particular about $\tau(n)$.

Conjecture 2 (Lehmer's Conjecture)**.** *For all $n \ge 1$, we have $\tau(n) \ne 0$.*

The first thing you can try to do is search for congruence obstructions.

Example 1. Let $A := \sum_{n\ge1} \sigma_3(n)q^n$, $B := \sum_{n\ge1}\sigma_5(n)q^n$, so that $E_4 = 1 + 240A$, $E_6 = 1 - 504B$. Then we have

$$\Delta = \frac{(1+240A)^3 - (1-504B)^2}{1728} = \frac{5(A-B)}{12} + B + 100A^2 - 147B^2 + 8000A^3.$$

The only denominator is a 12. But $12|(A-B)$ since $12|(\sigma_5(n) - \sigma_3(n))$ for all n as

$$12|(d^5 - d^3) = d^3(d^2-1)$$

(if d is even, then $4|d^3(d^2-1)$ and if d is odd then $d^2 \equiv 1 \pmod 4$ so $d^2-1 \equiv 0 \pmod 4$, and if $d \equiv 0 \pmod 3$, $d^3 \equiv 0 \pmod 3$, and if $d \equiv \pm 1 \pmod 3$, then $d^2 \equiv 1 \pmod 3$ so $d^2-1 \equiv 0 \pmod 3$)). In fact, we actually even have that $24|(A-B)$. Thus, we have that $\tau(n) \in \mathbb{Z}$, which is clear from the product formula, but not from our original formula. The above together with the claimed 24 divisibility (which I leave as an exercise) also implies that

$$\Delta \equiv B + B^2 \pmod 2.$$

By the Freshmen's Dream we have for any $a_n \in \mathbb{Z}$:

$$\left(\sum_n a_n q^n\right)^2 \equiv \sum_n a_n q^{2n}.$$

Thus,

$$\tau(n) \equiv \sigma_5(n) + \sigma_5(n/2) \pmod 2 \equiv \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Thus, $\tau(n)$ is odd half the time, and in particular non-zero at least half the time.

Even better, we can work in M_{12}, a two dimensional space, to directly check using only two coefficients that

$$G_{12} = \Delta + \frac{691}{156}\left(\frac{E_4^3}{720} + \frac{E_6^2}{1008}\right).$$

Thus, we have the surprising congruence

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}$$

(Ramanujan also conjectured this, but it was first proven by Watson). The 691 is really there because its the numerator of B_{12} and so in the constant term of G_{12}. This is a **very** special congruence, and Serre and Swinnerton-Dyer showed how this is really coming from a Galois representation in a deep way. Using the theory of Galois representations, there is the following brand new result

Theorem (Balakrishnan, Craig, Ono, May 2020)**.** *For $n \geq 1$, we have*

$$\tau(n) \notin \{\pm 1, \pm 3, \pm 5, \pm 7, \pm 691\}.$$

Using an explicit version of Sato-Tate, we also have the strong density result:

Theorem (Rouse-Thorner)**.** *Assume some standard conjectures, such as GRH but also conjectures which were just proven in late 2019 by Newton and Thorne, then we have*

$$\lim_{x\to\infty} \frac{\#\{n \leq x \mid \tau(n) \neq 0\}}{x} > 1 - 1.54 \cdot 10^{-13}$$