MODULAR FORMS LECTURE 14: DIFFERENTIAL OPERATORS AND QUASIMODULAR FORMS

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> To improve upon Hecke [in a treatment along classical lines of the theory of algebraic numbers] **would be a futile and impossible task**.
>
> André Weil

1. Differential operators

Previously, we saw that the Eisenstein series/Poincaré series construction for E_k fails at $k = 2$, but only just barely. We also saw that there are no non-zero modular forms of weight 2. However, E_2 is very nearly a modular form. Since we saw its q-expansion before, its translation invariant, and its S transformation is given by:

$$E_2|_2(S-1) = \frac{12}{2\pi i\tau}.$$

This "error" to modularity, instead of being 0 if it were a true modular form, is a simple rational function in τ. We will soon see where this formula comes from.

Corollary. *The non-holomorphic function*

$$E_2^*(\tau) := E_2(\tau) - \frac{3}{\pi v}$$

transforms like a modular form of weight 2*.*

Exercise 1. *Show that this follows from the transformation property of* E_2 *above.*

Remark. *This function has several interpretations. We will see one below. Another is that its the first example of a* **mock modular form***, which leads to the theory of harmonic Maass forms (real-analytic modular forms that are harmonic with respect to the hyperbolic metric instead of holomorphic).*

In proving the Valence Formula, we also saw that the derivative of a modular form just barely fails to be a modular form. In fact, its failure is essentially the same as that of E_2 above. The problem is that for functions $f\colon \mathbb{H} \to \mathbb{C}$, we don't have intertwining:

$$(Df)|_{k+2}\gamma \neq D(f|_k\gamma),$$

where $D := \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$.

However, there are two corrections to this failure.

Definition. The **Serre derivative** is given by

$$\vartheta_k(f) := Df - \frac{k}{12}E_2 f.$$

The **raising operator** is given by

$$R_k(f) := -4\pi D(f) + \frac{kf}{v}.$$

Remark. *If you read the literature on raising (and lowering) operators, there are different conventions.*

These operators "fix" modularity, and map from weight k modular forms to weight $k+2$ modular forms. For example, we have the intertwining property

$$(R_k f)|_{k+2}\gamma \neq R_k(f|k\gamma)$$

for any function $f\colon \mathbb{H} \to \mathbb{C}$ and any $\gamma \in \Gamma$.

Exercise 2. *Show that the Serre derivative and raising operatorssend modular forms of weight k to modular forms of weight $k+2$ (this is only about the transformation behavior; the raising operator destroys holomorphicity).*

You may have noticed that for $k = 0$, we have $R_0 = -4\pi D$, as the extra piece cancels out. Thus, differentiating something in weight 0 does give a modular form of weight 2. In general, its useful to have the exact relationship between iterated derivatives and raising operators.

Exercise 3. *Show that for all $k \in \mathbb{Z}$ and $n \geq 0$, we have*

$$R_k^n = \sum_{r=0}^{n} (-1)^r \binom{n}{r} (k+r)_{n-r} v^{r-n} (4\pi D)^r,$$

where $(a)_n := a(a+1)\dots(a+n-1)$ is a rising factorial. Do this by induction. [Note: This is a pretty hard exercise, as it uses some tricky identities. We wrote down the details of the proof in my book with Bringmann, Folsom, and Ono on harmonic Maass forms, and I can share the details of this computation with you if you're interested.]

Corollary (Bol's Identity)**.** *For $k \geq 1$, we have*

$$D^{k-1} = \frac{1}{(-4\pi)^{k-1}} R_{2-k}^{k-1},$$

where the iterated raising operator is

$$R_n^k := R_{k+2(n-1)} \circ \ldots \circ R_k.$$

That is, you compose n raising operators and raise the weight by 2 each time.

Proof. Using the result from the exercise, let $k \mapsto 2-k$ and plug in $n = k-1$. Then the terms $(2-k+r)_{k-1-r} = (2-k+r)(3-k+r)\cdots(-2)\cdot 0$ all vanish except for $r = n = k-1$, when we have the empty product which we consider to be 1. This term gives $(-1)^{k-1}\binom{k-1}{k-1} \cdot 1 \cdot v^0 (4\pi D)^{k-1}$, which is equivalent to the claim. □

Corollary. *The $k-1$-st derivative of a weight $2-k$ weakly holomorphic modular form is modular of weight k. For instance $D(M_0^!) \subseteq M_2^!$, and $D^3(M_{-2}^!) \subseteq M_4^!$.*

Proof. The operator D^{k-1} preserves holomorphicity. By Bol's Identity, it is also an iterated raising operator, and so preserves modularity. □

Remark. *There is something very special about the relationship between weights k and $2-k$. There are general reasons for this, including* **Serre duality***.*

2. Quasi and almost holomorphic modular forms

The definitions above give rise to two related spaces of forms.

Definition. An **almost holomorphic modular form** is a function which is a modular form but where the condition that it is holomorphic is replaced by the condition that it is a polynomial in $1/v$ with holomorphic coefficients. A **quasimodular form** is a holomorphic part of an almost holomorphic modular form.

We saw earlier that all modular forms are polynomials in E_4 and E_6. We have the following extension, which we'll state without proof.

Proposition. *The algebra of almost holomorphic modular forms $\mathcal{M}^*$ is $\mathbb{C}[E_2^*, E_4, E_6]$. The algebra of quasimodular forms $\widetilde{\mathcal{M}}$ is $\mathbb{C}[E_2, E_4, E_6]$.*

The algebra of quasimodular forms is closed under differentiation. In particular, differentiation has the following effect on the generators.

Exercise 4. *Show the* **Ramanujan formulas***:*

$$DE_2 = \frac{E_2^2 - E_4}{12},$$
$$DE_4 = \frac{E_2E_4 - E_6}{3},$$
$$DE_6 = \frac{E_2E_6 - E_4^2}{2}.$$

3. Quasimodularity of E_2

Let's finally prove the transformation formula for E_2. Since its just a multiple of G_2, this is equivalent to determining this for G_2.

Theorem. *For any $\gamma \in \Gamma$, we have*

$$G_2(\gamma \cdot \tau) = (c\tau + d)^2 G_2(\tau) - \pi i c(c\tau + d).$$

Proof. We use a common technique called the **Hecke trick**. We'll closely follow the exposition in Zagier's chapter of the "1-2-3 of modular forms". Recall that we can write

$$G_2(\tau) = \frac{1}{2} \sum_{m,n}{}' \frac{1}{(m\tau + n)^2}.$$

This doesn't converge absolutely, but nearly does. Fix this by deforming $\varepsilon > 0$:

$$G_{2,\varepsilon} := \frac{1}{2} \sum_{m,n}{}' \frac{1}{(m\tau + n)^2 |m\tau + n|^{2\varepsilon}}.$$

By the same proof as we did for the modularity of Eisenstein series before, this converges absolutely and we have

$$G_{2,\varepsilon}(\gamma\tau) = (c\tau + d)^2 |c\tau + d|^{2\varepsilon} G_{2,\varepsilon}(\tau).$$

We aim to show that
$$\lim_{\varepsilon\to 0^+} G_{2,\varepsilon}(\tau) = G_2(\tau) - \frac{\pi}{2v}.$$
If we know this, then we have that $G_2^*(\tau) = G_2(\tau) - \frac{\pi}{2v}$ is an almost holomorphic modular form of weight 2. Similarly to the exercise above, this easily implies the claimed transformation formula for $G_2(\tau)$.

To see this limit formula, let
$$I_\varepsilon(\tau) := \int_{-\infty}^{\infty} \frac{dt}{(\tau+t)^2|\tau+t|^{2\varepsilon}}, \qquad \left(\tau\in\mathbb{H},\ \varepsilon > -\frac{1}{2}\right).$$
Then for $\varepsilon > 0$, we have
$$G_{2,\varepsilon}(\tau) - \sum_{m\geq 1} I_\varepsilon(m\tau) = \sum_{n\geq 1}\frac{1}{n^{2+2\varepsilon}} + \sum_{m\geq 1}\sum_{n\in\mathbb{Z}}\left[\frac{1}{(m\tau+n)^2|m\tau+n|^{2\varepsilon}} - \int_n^{n+1}\frac{dt}{(m\tau+t)^2|m\tau+t|^{2\varepsilon}}\right].$$
Both sides converge absolutely and locally uniformly (in the second one, the term in big brackets is $O(|m\tau+n|^{-3-2\varepsilon})$). Thus the limit exists as $\varepsilon\searrow 0$. We can find this limit by plugging in $\varepsilon = 0$ term by term. Now $I_0(\tau) = 0$, and plugging in $\varepsilon = 0$ in the rest kills the absolute values and gives us an expression for G_2 (just with the $m = 0$ terms split off, just as we did when we computed the Fourier expansions of G_k).

We also compute, for any $\varepsilon > -\frac{1}{2}$, that $I_\varepsilon(\tau)$ doesn't depend on u:
$$I_\varepsilon(u+iv) = \int_{-\infty}^{\infty}\frac{dt}{(u+iv+t)^2((u+t)^2+v^2)^\varepsilon} = \int_{-\infty}^{\infty}\frac{dt}{(t+iv)^2(t^2+v^2)^\varepsilon} = \frac{I(\varepsilon)}{v^{1+2\varepsilon}},$$
where
$$I(\varepsilon) := \int_{-\infty}^{\infty}\frac{dt}{(t+i)^2(t^2+1)^\varepsilon}.$$
Thus,
$$\sum_{m\geq 1} I_\varepsilon(m\tau) = \frac{I(\varepsilon)\zeta(1+2\varepsilon)}{v^{1+2\varepsilon}}.$$
Clearly $I(0) = 0$, and
$$I'(0) = -\int_{-\infty}^{\infty}\frac{\log(t^2+1)dt}{(t+i)^2} = \left[\frac{1+\log(t^2+1)}{t+i} - \arctan(t)\right]_{-\infty}^{\infty} = -\pi.$$
Further,
$$\zeta(1+2\varepsilon) = \frac{1}{2\varepsilon} + O(1)$$
($\zeta(s)$ has a simple pole of residue 1 at $s = 1$), and so
$$\lim_{\varepsilon\to 0}\frac{I(\varepsilon)\zeta(1+2\varepsilon)}{v^{1+2\varepsilon}} = -\frac{\pi}{2v}.$$
□

Exercise 5. *Fill in any details in the sketch above.*