MODULAR FORMS LECTURE 11: FIRST CONSTRUCTIONS: EISENSTEIN SERIES

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> For fifteen days I strove to prove that there could not be any functions like those I have since called Fuchsian functions. I was then very ignorant; every day I seated myself at my work table, stayed an hour or two, tried a great number of combinations and reached no results. One evening, contrary to my custom, I drank black coffee and could not sleep. Ideas rose in crowds; I felt them collide until pairs interlocked, so to speak, making a stable combination. By the next morning I had established the existence of a class of Fuchsian functions, those which come from the hypergeometric series; I had only to write out the results, which took but a few hours.
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> Poincaré on his discoveries on what we'd now call modular forms.

1. Constructing modular forms

We have seen properties of modular forms. As Poincaré asks above, do any interesting ones exist? The answer is yes, of course! Let's give our first examples. Last time, we saw that modular symmetry is the same thing as being invariant under the Petersson slash action. So the group Γ acts on the set of functions on the upper half plane, and we want to find invariant functions which are analytically nice and don't grow too fast.

We saw another instance of this type of question in the elliptic function notes. Namely, what we can try to do is take a **group average**. This is called the method of **Poincaré series**, and we will return to it throughout the class. It is one of the two primary methods of producing modular forms from scratch. Given a **seed function** $\varphi(\tau)$, the ansatz is to consider a sum of the form

$$P_k(\varphi;\tau) := \sum_{\gamma\in\Gamma} \varphi|_k\gamma(\tau).$$

Then, exactly as we saw with elliptic functions, $P_k(\varphi;\tau)$ will be modular if we can guarantee that φ is nice enough and has good enough growth properties, and that this sum is absolutely convergent. The first special case we'll consider is what happens when $\varphi(\tau)$ is a constant function. If, say, $\varphi(\tau)=1$, then we'd want to try

$$P_k(1;\tau) := \sum_{\gamma\in\Gamma} 1|_k\gamma(\tau) = \sum_{\gamma\in\Gamma} \frac{1}{(c\tau+d)^k}.$$

If k is odd, there are no non-zero modular forms, so let's assume k is even. At face value this will not work. For instance, if $c\tau+d=\pm1$, namely $\gamma\in\Gamma_\infty = \left\{\pm\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right) \;\middle|\; n\in\mathbb{Z}\right\}$, the **stabilizer of infinity**, then we'll get another copy of 1 in the sum. Thus, this sum has infinitely many copies of "1" in it, so it diverges. However, we can easily fix this.

Definition. The **Eisenstein series** of weight k is

$$E_k(\tau) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} 1_k|\gamma(\tau) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \frac{1}{(c\tau+d)^k} = \frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}} \frac{1}{(c\tau+d)^k}.$$

Exercise 1. *Check the last equality (hint: check that two matrices have the same bottom row iff you can transform one into the other by multiplying by an element of Γ_∞, and check that the set of possible bottom rows is the set of pairs of coprime integers; where does the factor $\frac{1}{2}$ come from?).*

Theorem. *We have that $E_k(\tau)\in M_k$ for any $k\geq 4$ even. In particular, such M_k are not zero-dimensional.*

Proof. For modularity, we just have to check the absolute convergence properties. This is either a very similar calculation as the previous exercise we sketched in class, or directly follows from it since we saw this same function as a Taylor coefficient of the Weierstraß $\wp$-function. Note that we called the Eisenstein series there G_k, while here we called them E_k. That is because we have taken a different normalization here. So to directly apply our previous result, you must check that if

$$G_k(\tau) := \frac{1}{2}\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m\tau+n)^k}$$

then

$$G_k(\tau) = \zeta(k)E_k(\tau),$$

where $\zeta(k) := \sum_{n\geq1} n^{-k}$ is the **Riemann zeta function** (hint: this is because any pair $(m,n)\in\mathbb{Z}^2$ is uniquely of the form $r(c,d)$ with $(c,d)=1$ and $r>0$). To check the absolute convergence directly, you basically just need to note that the inputs (c,d)

with $N \le |c\tau + d| < N+1$ all correspond to lattice points $c\tau + d \in \mathbb{Z}\tau + \mathbb{Z}$ in the annulus of radii N and $N+1$. The area of this annulus is $\pi(N+1)^2 - \pi N^2$, and the number of lattice points must be $O(N)$ (the constant multiple of N depends on the area of the fundamental parallelogram). So the sum behaves at worst like $\sum_{N\ge 1} \frac{N}{N^k}$, which for $k > 2$, is absolutely convergent.

It is also automatically holomorphic on $\mathbb{H}$ once we have these convergence properties. We need to check that its holomorphic at $i\infty$. This holds since it is bounded as taking $\tau = iv$ and letting $v \to \infty$:

$$\lim_{v\to\infty} \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)=1}} \frac{1}{(civ+d)^k} = \frac{1}{2} \sum_{\substack{c=0,d\in\mathbb{Z} \\ (c,d)=1}} \frac{1}{d^k} = \frac{1}{2} \sum_{c=0,d=\pm 1} \frac{1}{d^k} = 1.$$

Thus, $E_k(\tau) = 1 + O(q)$, and we are done. $\square$

Anytime we get a modular form, our first question is what its Fourier expansion is.

Theorem 1.1. *For even $k \ge 4$, the Fourier expansion of E_k is given by*

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n\ge 1} \sigma_{k-1}(n) q^n \in \mathbb{Q}[[q]].$$

Here, B_k is a **Bernoulli number***, defined by the (exponential) generating function*

$$\frac{x}{e^x - 1} = \sum_{k\ge 0} B_k \frac{x^k}{k!}.$$

Proof. A famous theorem from complex analysis is the infinite product formula for the sine function:

$$\sin(\pi\tau) = \pi\tau \prod_{n\ge 1} \left(1 - \frac{\tau^2}{n^2}\right) = \pi\tau \prod_{n\ge 1} \left(1 - \frac{\tau}{n}\right)\left(1 + \frac{\tau}{n}\right)$$

(it can be shown using the Weierstraß factorization formula, and it has to do with the fact that $\sin(\pi\tau)$ has a simple zero at all $n \in \mathbb{Z}$, so like with polynomials you can try to build up functions by their roots). Taking the log derivative of this formula turns the product into a sum and gives

$$\pi\cot(\pi\tau) = \frac{1}{\tau} + \sum_{n\ge 1} \left(\frac{-\frac{1}{n}}{1 - \frac{\tau}{n}} + \frac{\frac{1}{n}}{1 + \frac{\tau}{n}} \right) = \frac{1}{\tau} + \sum_{n\ge 1} \left(\frac{1}{\tau - n} + \frac{1}{\tau + n} \right).$$

For short hand we will write this as

$$\pi\cot(\pi\tau) = \sum_{n\in\mathbb{Z}} \frac{1}{(\tau + n)}.$$

This sum is not absolutely convergent, but it is interpreted as a **Cauchy principal value**; we have to let the negative and positive terms in the sum tend to infinity at roughly the same rate. That is, we take a sum from $n = -M$ to $n = N$ with $|N - M|$ bounded and take a limit as $M, N \to \infty$.

We can use complex exponentials and geometric series to write the LHS as

$$\pi i \frac{e^{\pi i\tau} + e^{-\pi i\tau}}{e^{\pi i\tau} - e^{-\pi i\tau}} = -\pi i \frac{1+q}{1-q} = -2\pi i \left(\frac{1}{2} + \sum_{j\geq 1} q^j \right).$$

Meanwhile, the LHS of the above looks like something that would be in the Eisenstein series sum, but without powers in the denominator. Of course, we can access higher powers by repeatedly differentiating. Specifically, we find (note: keep in mind the useful formula: $\frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$)

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^k} = \frac{(-1)^{k-1}}{(k-1)!}\frac{d^{k-1}}{d\tau^{k-1}} \left(-2\pi i \left(\frac{1}{2} + \sum_{j\geq 1} q^j \right) \right) = \frac{(-2\pi i)^k}{(k-1)!} \sum_{j\geq 1} j^{k-1} q^j.$$

Now we are ready to compute the Fourier expansion of E_k. Since E_k is the same as G_k up to a constant, we can work with G_k from above. We simply split off the terms with $m = 0$ off from the rest and compute:

$$G_k(\tau) = \frac{1}{2} \sum_{0\neq n\in\mathbb{Z}} \frac{1}{n^k} + \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z} \\ m\neq 0}} \frac{1}{(m\tau+n)^k} = \sum_{n\geq 1} \frac{1}{n^k} + \sum_{m\geq 1}\sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^k}$$

$$= \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{m\geq 1}\sum_{r\geq 1} r^{k-1} q^{mr} = \frac{(2\pi i)^k}{(k-1)!} \left(-\frac{B_k}{2k} + \sum_{n\geq 1} \sigma_{k-1}(n) q^n \right).$$

Here we used Euler's formula

$$\zeta(k) = -\frac{(2\pi i)^k B_k}{2k!}.$$

Since this formula is so famous, its worth giving it a short proof. From the above write

$$\pi i\tau \frac{e^{\pi i\tau} + e^{-\pi i\tau}}{e^{\pi i\tau} - e^{-\pi i\tau}} = \pi i\tau + \frac{2\pi i\tau}{e^{2\pi i\tau} - 1} = \frac{x}{2} + \frac{x}{e^x - 1},$$

where we have set $x := 2\pi i\tau$. Our cotangent identity then implies that

$$\frac{x}{2} + \frac{x}{e^x - 1} = \sum_{n\in\mathbb{Z}} \frac{x}{x + 2\pi i n} = 1 + \sum_{0\neq n\in\mathbb{Z}} \sum_{k\geq 1} (-1)^{k+1} \frac{x^k}{(2\pi i n)^k}$$

$$= 1 + \sum_{k\geq 1} (-1)^{k+1} \left(\frac{x}{2\pi i} \right)^k \sum_{0\neq n\in\mathbb{Z}} \frac{1}{n^k} = 1 - 2 \sum_{\substack{k\geq 0 \\ k \text{ even}}} \zeta(k) \left(\frac{x}{2\pi i} \right)^k.$$

Now the coefficient of x^k for $k \geq 2$ even in the first expression is by definition $\frac{B_k}{k!}$, while the same coefficient of x^k on the last expression is $-\zeta(k)(2\pi i)^{-k}$.

By dividing by $\zeta(k)$, we obtain the desired Fourier expansions for E_k. $\square$

Example 1. The first few Eisenstein series from above are:

$$E_4(\tau) = 1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n,$$
$$E_6(\tau) = 1 - 504 \sum_{n\geq 1} \sigma_5(n) q^n,$$
$$E_8(\tau) = 1 + 480 \sum_{n\geq 1} \sigma_7(n) q^n.$$

Previously from elliptic functions we showed that $E_4^2 = E_8$ (after translating from G's to E's). We will see another reason for this shortly. Plugging these expansions into the previous discussion recovers our claimed combinatorial convolution identities.

Question. *What happens when $k = 2$?*

A careful study of the proof above shows that for $k = 2$, we are just on the edge of absolute convergence. That is, the sums above absolutely converge for all $k > 2$. Since we are so close, can modularity be saved? The answer is yes! This is analogous to use saving the elliptic function we tried to build with a pole of order 2 at $z = 0$; a slight modification fixes the convergence issue. This is a powerful technique we'll study soon. The short answer is that the natural guess

$$E_2(\tau) = 1 - 24 \sum_{n\geq 1} \sigma_1(n) q^n$$

is nearly a modular form, and our first example of a **quasimodular form**.