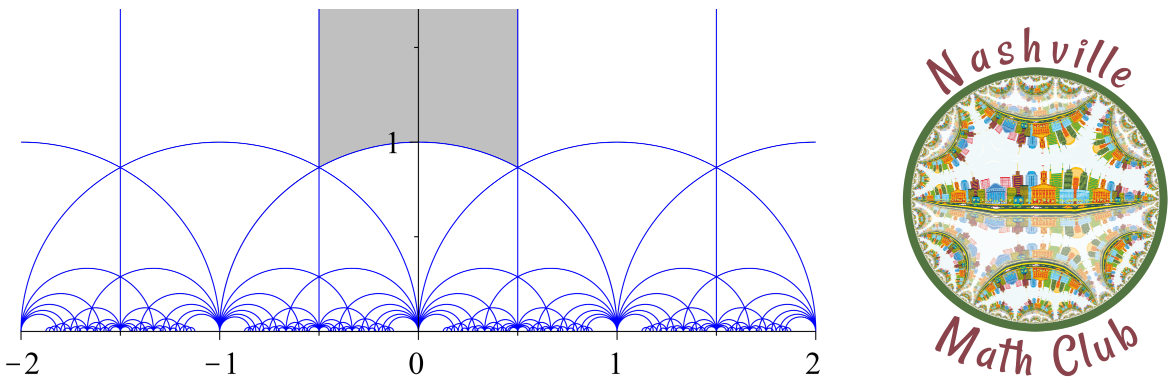
MODULAR FORMS LECTURE 10: MODULAR FORMS: THE FUNDAMENTAL DOMAIN

LARRY ROLEN, VANDERBILT UNIVERSITY, FALL 2020



Tiling of upper half plane by Kilom691 and Alexander Hulpke [CC-BY-SA-4.0], and of the circle with hyperbolic polygons of Nashville by me.

0.1. **Slash action and automorphy factors.** We have seen the definitions of different types of modular forms, as well as some of their basic properties. We now continue with another point of view on modularity which will be frequently useful.

Definition. The **Petersson slash action** of Γ on the set of functions $f \colon \mathbb{H} \to \mathbb{C}$ is given (in weight k) by

$$f|_k\gamma(\tau) = f|\gamma(\tau) := f(\gamma \cdot \tau)(c\tau + d)^{-k}.$$

Thus,

$$f \text{ is modular of weight } k \iff f|_k(\gamma) = f \text{ for all } \gamma \in \Gamma.$$

Exercise 1. *Check that this is indeed a group action.*

There is another special property of $(c\tau + d)^k$. It is what we call an **automorphy factor**. This means the following. It is standard to denote an automorphy factor of this by j, so let's set

$$j(\gamma, \tau) := (c\tau + d)^k.$$

If f is modular, we have

$$f(\gamma\tau) = j(\gamma, \tau)f(\tau)$$

for all $\gamma \in]\Gamma$, and so in particular

$$\frac{f(\alpha\beta\tau)}{f(\tau)} = \frac{f(\alpha\beta\tau)}{f(\beta\tau)} \cdot \frac{f(\beta\tau)}{f(\tau)}.$$

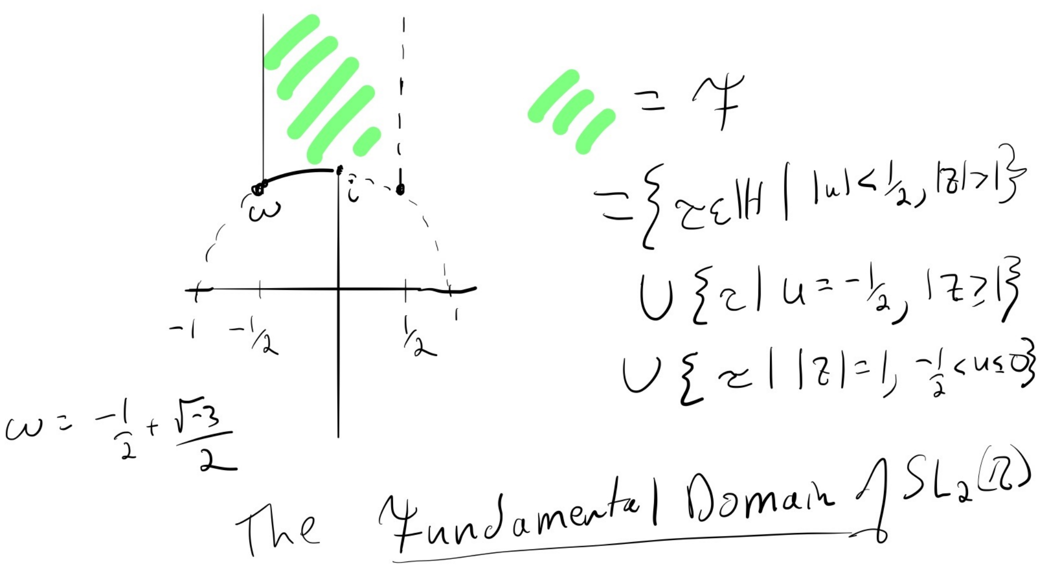
Thus, we must have that

$$j(\alpha\beta, \tau) = j(\alpha, \beta\tau) j(\beta, \tau).$$

This consistency relation is what we mean by an automorphy factor, and is quite subtle. For $k \in \mathbb{Z}$, $j(\gamma, \tau)$ satisfies this, but for $k \notin \mathbb{Z}$, this can fail! In particular, we will care quite a lot about $k \in \frac{1}{2} + \mathbb{Z}$ later (its needed to solve the Congruent Number Problem), and so the definition of modular forms in that case must be modified to fix this consistency condition first.

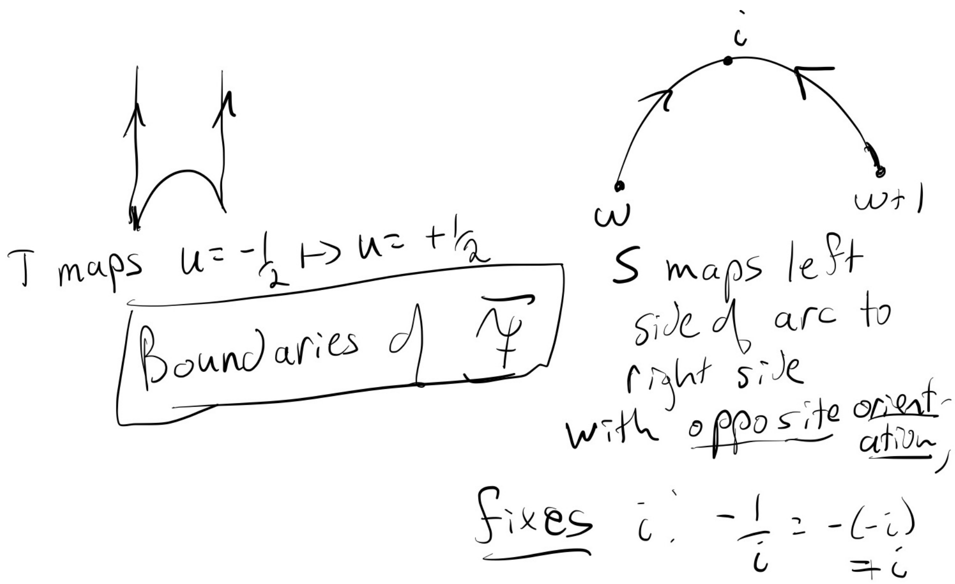
0.2. **The Fundamental Domain.** Just like parallelograms were fundamental domains for the action of a lattice Λ on $\mathbb{C}$ by translation, there will be a nice fundamental domain for the action of Γ on $\mathbb{H}$ by fractional linear transformations. Recall that this means it will be a set where every point is equivalent to exactly one point inside of the fundamental domain under a fractional linear transformation.

Let's draw the famous picture and then prove its correct.



In short, the interior is the region bounded by the lines $u = \pm\frac{1}{2}$ and the unit circle, but you have to be careful about the boundary so that equivalent points aren't represented more than once. In the hyperbolic geometry, this is a **hyperbolic triangle**, and the two sides meet at $i\infty$. The points ω, i will play a special role, as we'll soon see.

Before showing this is a fundamental domain, let's first discuss what actually happens on the boundary, and why we exclude some points on the boundary of the closure of $\mathcal{F}$.



Theorem 0.1. *The region $\mathcal{F}$ is a fundamental domain for $\Gamma \circlearrowright \mathbb{H}$.*

Proof. Let $\tau \in \mathbb{H}$. First we show we can move it into Γ. We saw before that

$$\operatorname{Im}(\gamma\tau) = \frac{u}{|c\tau + d|^2}.$$

The values $c\tau + d$ range over $\mathbb{Z}\tau + \mathbb{Z}$, a **lattice**. Thus, the non-zero values are bounded away from zero (there is some non-zero lattice vector of minimal length; lattices are discrete). Thus, there is some γ which makes $|c\tau + d|$ minimal, and hence makes $\operatorname{Im}(\gamma\tau)$ **maximal**. Shifting by powers of T you can then force the point to satisfy

$$-\frac{1}{2} \leq \operatorname{Re}(T^n\gamma\tau) < \frac{1}{2}.$$

Thus, we can assume that $\gamma\tau$ is in the strip $-\frac{1}{2} \leq u < \frac{1}{2}$.

If $\gamma\tau \notin \mathcal{F}$, then $|\gamma\tau| < 1$ (unless its on the right arc of the unit circle which we excluded, but then just apply S to get in $\mathcal{F}$). But then

$$\operatorname{Im}(S\gamma\tau) = \frac{\operatorname{Im}(\gamma\tau)}{|\gamma\tau|^2} > \operatorname{Im}(\gamma\tau),$$

a *contradiction*. Thus, $\gamma\tau \in \mathcal{F}$, and we're done.

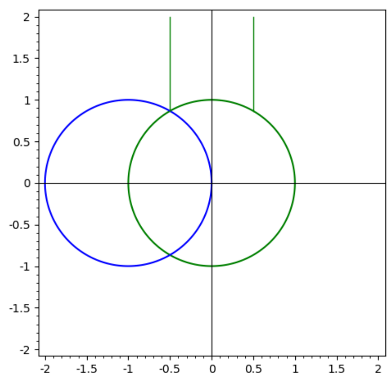
We now need to show **uniqueness**. We already drew pictures of what happens on the boundaries, and carefully looking at those and the definition of $\mathcal{F}$ shows that no two boundary points are Γ-equivalent. Now let's look at the interior. Suppose that $\tau_1, \tau_2 \in \mathcal{F}^\circ$ are Γ-equivalent. WLOG, say $v_2 \geq v_1$, and say $\tau_2 = \gamma\tau_1$. By our formula for $\operatorname{Im}(\gamma\tau)$, we have $|c\tau_1 + d| \leq 1$.

Since $\tau_1 \in \mathcal{F}^\circ$, this is impossible if $|c| \geq 2$. This leaves several possibilities.

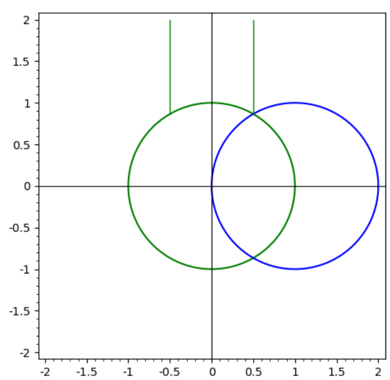
Case i). $c = 0, d = \pm 1$: This is impossible as $\gamma = T^m$ then but this takes points outside of $\mathcal{F}$.

Case ii). $c = \pm 1, d = 0$: Then $|\pm\tau_1| = |\tau_1| \leq 1$ and so τ_1 is on the unit circle. But we've already talked about boundary points.

Case iii). $c = d = \pm 1$: Then $|\tau_1 + 1| \leq 1$, then τ_1 is at most distance 1 away from $(-1, 0)$. As the following picture illustrates, the only point inside of the Fundamental Domain satisfying this is ω, which is again not in the interior.



Case iv). $c = -d = \pm 1$: Just as in the last case, we find that $\tau_1 = \omega + 1$ is not in the interior, as illustrated by the following picture:



□

We will also be interested in the **stabilizers** of points; the subgroup of Γ consisting of those matrices which fix a point. There are always two trivial points in the stabilizer, namely $\pm I$. Usually, this is it. Similar calculations as in the last proof give the following.

Theorem 0.2. *The stabilizer of a point $\tau \in \mathcal{F}$ is given by:*

$$\Gamma_\tau = \begin{cases} \{\pm I\}, & \text{if } \tau \neq i, \omega \\ \{\pm I, \pm S\} & \text{if } \tau = i, \\ \{\pm I, \pm ST, \pm (ST)^2\} & \text{if } \tau = \omega. \end{cases}$$

In particular, the order of the stabilizer is either $2, 4$*, or* 6 *depending on these cases.*

Exercise 2. *Prove this.*