Mock modular Eisenstein series and holomorphic projection

Larry Rolen

Vanderbilt University

Classical Eisenstein series

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The case of weight 2

- The series $G_2(\tau)$ is not modular, but rather the non-holomorphic "correction"

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- E_2/G_2 is the first example of a **mock modular form**.

Motivating question

Question

Are there similar formulas for other types of mock modular forms? Is there a family of simple combinatorial mock modular forms?

Class number generating function

- Next simplest mock modular form: $\mathcal{H}(\tau) := \sum_{n\geq 0} H(n)q^n$, where $H(n)$ is the *Hurwitz class number*, $H(0) := -1/12$.

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Theorem (Zagier)

$$\widehat{\mathcal{H}}(\tau) := \mathcal{H}(\tau) + \frac{1}{4\sqrt{\pi}} \sum_{n\geq 1} n\Gamma\left(-\frac{1}{2}, 4\pi n^2 \operatorname{Im}(\tau)\right) q^{-n^2} + \frac{1}{8\pi\sqrt{\operatorname{Im}(\tau)}}$$

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- Hurwitz-Kronecker class number relation:

$$\sum_{m\in\mathbb{Z}} H(4n - m^2) = 2\sigma_1(n) - \sum_{d|n} \min(d, n/d).$$

The method of Poincaré series

- Petsersson slash action:

$$f|_k\gamma := (c\tau + d)^{-k} f(\gamma \cdot \tau), \qquad \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

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- Petersson coefficient formula for $f \in S_k$:

$$\langle f, P_{k,m} \rangle \doteq [q^m] f(\tau).$$

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- Then the following is modular of weight k:

$$\pi_{\text{hol}}(f) := \sum_n a(n)q^n.$$

Sketch of proof of class number relation

- Let $f(\tau) := \widehat{\mathcal{H}}(4\tau) \cdot \theta(\tau)$, with Jacobi's $\theta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}$.

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- Special function identity:

$$\int_0^\infty \Gamma(1-k,4\pi|m|v)e^{-4\pi nv}v^{k+\ell-2}dv = \frac{(4\pi|m|)^{k-1}\Gamma(\ell)}{(k+\ell-1)(4\pi(n-m))^\ell}\,{}_2F_1(1,\ell,k+\ell;n/(n-m)).$$

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Then compute $\pi_{\mathrm{hol}}(f) = \frac{-1}{12}E_2(\tau)$.

Recursions for mock theta functions

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- These are analogous to Conway-Norton's theory of *replicable functions*, and were crucial in the proof of Duncan, Griffin, and Ono of the Umbral Moonshine Conjecture.

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$$\sigma_{\psi}^{\mathrm{sm}}(n) := \sum_{\substack{d \mid n \\ 1 \leq d \leq n/d \\ d \equiv n/d \pmod 2}} \psi\left(\frac{(n/d)^2 - d^2}{4}\right) d.$$

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- The mock Eisenstein series is

$$\mathcal{E}_\psi(\tau) := \frac{1}{\theta_\psi(\tau)} \cdot \sum_{n\geq 1} \sigma_\psi^{\mathrm{sm}}(n) q^n.$$

Main theorem

Theorem (Mertens-Ono-R.)

The mock Eisenstein series $\mathcal{E}_\psi$ is a weight $1+\psi(-1)/2 \in \{1/2, 3/2\}$ mock modular form with (possible) poles on the upper half plane. Its shadow is (a multiple of) the Shimura theta function θ_ψ.

Applications to congruences

Theorem (Mertens-Ono-R.)

For any prime p and any $a, b \in \mathbb{N}$, there is a weight 2 modular form $F_{a,b}$ such that

$$\left(\theta_\psi(p^{2a}\tau)\mathcal{E}_\psi(\tau)\right) \Big| U(p^b) \equiv F_{a,b} \pmod{p^{\min(a,b)}}.$$

Example 1

- When $\psi = \chi_{12} := \left(\frac{12}{\cdot}\right)$, then

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- Our theorem above recovers known congruences and p-adic properties due to Andrews-Garvan, Ahlgren-Kim, and Belmont-Lee-Musat-Trebat-Leder.

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- Then
$$\theta_{\chi_2}(\tau) \cdot \mathcal{E}_{\chi_2}(\tau) = 2 \sum_{n \geq 1} (-1)^n \mathrm{cpt}(n) q^{8n}.$$

Example 3

- The function

$$\frac{24\theta_{\chi_{-4}} \cdot \mathcal{E}_{\chi_{-4}}(\tau/8) - E_2(\tau)}{\eta(\tau)^3}$$

was studied by Eguchi-Taormina and Eguchi-Ooguri-Taormina-Yang in relation to the elliptic genus, and has been important in Mathieu Moonshine, $\mathrm{SO}(3)$ Donaldson invariants of $\mathbb{C}P^2$, and in Dabholkar-Murthy-Zagier's work on quantum black holes.