Zeta-polynomials for modular form periods

Larry Rolen

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Riemann’s zeta-function

**Definition (Riemann)**

For $\text{Re}(s) > 1$, define the **zeta-function** by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$
Riemann’s zeta-function

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Theorem (Fundamental Theorem)
1. The function \( \zeta(s) \) has an analytic continuation to \( \mathbb{C} \) (apart from a simple pole at \( s = 1 \) with residue 1).
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1. The function \( \zeta(s) \) has an analytic continuation to \( \mathbb{C} \) (apart from a simple pole at \( s = 1 \) with residue 1).

2. We have the functional equation

\[
\pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \zeta(s) = \pi^{-\frac{(1-s)}{2}} \cdot \Gamma \left( \frac{1-s}{2} \right) \cdot \zeta(1-s).
\]
$1$ million prize problem

Conjecture (Riemann)

Apart from the negative evens, the zeros of $\zeta(s)$ satisfy $\text{Re}(s) = \frac{1}{2}$. 

Remarks
1. The "line of symmetry" for $s \neq \frac{1}{2}$ is $\text{Re}(s) = \frac{1}{2}$.
2. The first "gazillion" zeros satisfy RH (Odlyzko).
3. Over 40% of the zeros satisfy RH (Selberg, Levinson, Conrey).
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The values $\zeta(-n)$

**Theorem (Euler)**

*As a power series in $t$, we have*

\[
\frac{t}{1-e^{-t}} = 1 + \frac{1}{2}t - t \sum_{n=1}^{\infty} \zeta(-n) \cdot \frac{t^n}{n!}.
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*Remark*

This series is essentially the generating function for $K$-groups of $\mathbb{Q}$. 
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$$\frac{t}{1 - e^{-t}} = 1 + \frac{1}{2} t - t \sum_{n=1}^{\infty} \zeta(-n) \cdot \frac{t^n}{n!}.$$ 

**Remark**

This series is essentially the generating function for $K$-groups of $\mathbb{Q}$. 
Definition (Manin)

A polynomial $Z(s)$ is a **zeta-polynomial** if it satisfies:
Manin’s Theory of Zeta-polynomials

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- For $s \in \mathbb{C}$ we have $Z(s) = \pm Z(1 - s)$. 

The values $Z(n)$ have a “nice” generating function.

The values $Z(n)$ encode arithmetic-geometric information.
Manin’s Theory of Zeta-polynomials

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A polynomial \( Z(s) \) is a zeta-polynomial if it satisfies:

1. It is arithmetic-geometric in origin.
2. For \( s \in \mathbb{C} \) we have \( Z(s) = \pm Z(1 - s) \).
3. If \( Z(\rho) = 0 \), then \( \text{Re}(\rho) = 1/2 \).
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- If $Z(\rho) = 0$, then $\text{Re}(\rho) = 1/2$.
- The values $Z(-n)$ have a “nice” generating function.
Introduction and Statement of Results

Manin’s Theory of Zeta-polynomials

**Definition (Manin)**

A polynomial $Z(s)$ is a **zeta-polynomial** if it satisfies:

- It is arithmetic-geometric in origin.
- For $s \in \mathbb{C}$ we have $Z(s) = \pm Z(1 - s)$.
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- The values $Z(−n)$ have a “nice” generating function.
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Manin’s Conjecture
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Conjecture (Manin)

*There is a theory of zeta-polynomials for modular form periods.*
Manin’s Conjecture

Conjecture (Manin)

There is a theory of zeta-polynomials for modular form periods.

Theorem (Main Theorem)

Manin’s Conjecture is true.
Theorem (Hecke, Atkin-Lehner, Shimura, Manin, and others)

If $f \in S_k(\Gamma_0(N))$ is a newform, then the following are true:

1. $L(f, s)$ has an analytic continuation to $\mathbb{C}$.
2. If $\gamma(f, s) := \prod_{p \mid N} \gamma_p(s)$, then $\Theta(f) \in \{\pm 1\}$ for which $\gamma(f, s) = \Theta(f) \cdot \gamma(f, k s)$.
3. There are numbers $\mathfrak{n}_{\pm f}$ such that for $1 \leq j \leq k - 1$:
   
   $L(f, j) \equiv \mathfrak{n}_{\pm f} \cdot (2\pi i)^j$. 


Fundamental Theorem for modular $L$-functions

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If $f \in S_k(\Gamma_0(N))$ is a newform, then the following are true:

1. $L(f, s)$ has an **analytic continuation** to $\mathbb{C}$.

2. If $\Lambda(f, s) := \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(f, s)$, then $\exists \varepsilon(f) \in \{\pm 1\}$ for which

$$\Lambda(f, s) = \varepsilon(f) \cdot \Lambda(f, k - s).$$
Fundamental Theorem for modular $L$-functions

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   $$\Lambda(f, s) = \epsilon(f) \cdot \Lambda(f, k - s).$$

3. There are numbers $\omega_f^{\pm}$ such that for $1 \leq j \leq k - 1$

   $$L(f, j) \in \mathbb{Q} \cdot (2\pi i)^j \cdot \omega_f^{\pm}.$$
Critical Values and Weighted Moments

Definition (Manin, Shimura)

If \( f \in S_k(\Gamma_0(N)) \) is a newform, then its critical \( L \)-values are

\[
\{L(f, 1), \ L(f, 2), \ L(f, 3), \ldots, \ L(f, k - 1)\}.
\]
Critical Values and Weighted Moments

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$$\{L(f, 1), \ L(f, 2), \ L(f, 3), \ldots, \ L(f, k - 1)\}.$$

**Definition (Ono-R-Sprung)**

If $m \geq 1$, then we define the **weighted moments**

$$M_f(m) := \frac{1}{(k - 2)!} \sum_{j=0}^{k-2} \binom{k-2}{j} \Lambda(f, j+1) \cdot j^m.$$
The zeta-polynomials \((k \geq 4\text{ even})\)
The zeta-polynomials \((k \geq 4\ \text{even})\)

**Definition (Ono-R-Sprung)**

The **zeta-polynomial** for \(f\) is

\[
Z_f(s) := \sum_{h=0}^{k-2} (-s)^h \sum_{m=0}^{k-2-h} \binom{m + h}{h} \cdot s(k - 2, m + h) \cdot M_f(m),
\]
The zeta-polynomials ($k \geq 4$ even)

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where the (signed) Stirling numbers of the first kind are given by

$$(x)_n = x(x-1)(x-2) \cdots (x-n+1) =: \sum_{m=0}^{n} s(n, m)x^m.$$
The $s(n, k)$ form Pascal-type triangles

We have the recurrence

$$s(n, k) = s(n - 1, k - 1) - (n - 1) \cdot s(n - 1, k).$$
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\[
\begin{array}{cccccc}
 & & & & & 1 \\
 & & & 0 & & 1 \\
 & & 0 & -1 & 1 & \\
 & 0 & 2 & -3 & 1 & \\
 0 & -6 & 11 & -6 & 1 & \\
 0 & 24 & -50 & 35 & -10 & 1 \\
 0 & -120 & 274 & -225 & 85 & -15 & 1 \\
\end{array}
\]

**Remark**

$Z_f(s)$ is a cobbling of layers of these weighted by moments $M_f(m)$. 
Theorem 1 (Ono-R-Sprung)

If $f \in S_k(\Gamma_0(N))$ is an even weight $k \geq 4$ newform, then we have:

1. For all $s \in \mathbb{C}$ we have that $Z_f(s) = (f) Z_f(1-s)$.
2. If $Z_f(\zeta) = 0$, then $\Re(\zeta) = 1/2$.

Remark: To completely obtain Manin's theory, we must show:

- The values $Z_f(n)$ have a "nice" generating function.
- The $Z_f(n)$ encode arithmetic-geometric information.
Theorem 1 (Ono-R-Sprung)

If \( f \in S_k(\Gamma_0(N)) \) is an even weight \( k \geq 4 \) newform, then we have:

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To completely obtain Manin’s theory, we must show:

- The values \( Z_f(-n) \) have a “nice” generating function.
Zeta-polynomials for modular form periods

Introduction and Statement of Results

Functional Equations and the Riemann Hypothesis

Theorem 1 (Ono-R-Sprung)

If \( f \in S_k(\Gamma_0(N)) \) is an even weight \( k \geq 4 \) newform, then we have:

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To completely obtain Manin’s theory, we must show:

- The values \( Z_f(-n) \) have a “nice” generating function.
- The \( Z(-n) \) encode arithmetic-geometric information.
Example of $\Delta \in S_{12}$

$$Z_\Delta(s) \approx (5.11 \times 10^{-7})s^{10} + \cdots - 0.0199s + 0.00596.$$
Example of $\Delta \in S_{12}$

\[ Z_\Delta(s) \approx (5.11 \times 10^{-7})s^{10} + \cdots - 0.0199s + 0.00596. \]
Theorem 2 (Ono-R-Sprung)

Define the \textbf{normalized period polynomial} for $f$ by

$$R_f(z) := \sum_{j=0}^{k-2} \binom{k-2}{j} \cdot \Lambda(f, k - 1 - j) \cdot z^j.$$
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Then we have that

$$\frac{R_f(z)}{(1 - z)^{k-1}} = \sum_{n=0}^{\infty} Z_f(-n) z^n.$$
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Then we have that

$$\frac{R_f(z)}{(1 - z)^{k-1}} = \sum_{n=0}^{\infty} Z_f(-n)z^n.$$ 

Remark (Euler)

$$\frac{t}{1 - e^{-t}} = 1 + \frac{1}{2} t - t \sum_{n=1}^{\infty} \zeta(-n) \cdot \frac{t^n}{n!}.$$
**Conjecture** (Bloch–Kato). Let $0 \leq j \leq k - 2$, and assume $L(f, j + 1) \neq 0$. Then we have

$$\frac{L(f, j + 1)}{(2\pi i)^{j+1} \Omega(-1)^{j+1}} = u_{j+1} \times \frac{\text{Tam}(j + 1)\#\text{III}(j + 1)}{\#H_Q^0(j + 1)\#H_Q^0(k - 1 - j)} =: C(j + 1)$$
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$$

Corollary (Ono-R-Sprung)

Assuming the Bloch-Kato Conjecture, we have that

$$
M_f(m) = \sum_{0 \leq j \leq k-2} C(j + 1)j^m.
$$
Combinatorial Polynomials $H_k^\pm (s)$

Definition (Binomial Coefficient)

If $x, y \in \mathbb{C}$, then the complex binomial coefficient $\binom{x}{y}$ is

$$\binom{x}{y} := \frac{\Gamma(x + 1)}{\Gamma(y + 1)\Gamma(x - y + 1)}.$$
### Combinatorial Polynomials $H^\pm_k(s)$

#### Definition (Binomial Coefficient)
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$$

#### Definition (Special Polynomials)
If $k \geq 4$ is even, then

$$
H^+_k(s) := \binom{s + k - 2}{k - 2} + \binom{s}{k - 2},
$$
$$
H^-_k(s) := \sum_{j=0}^{k-3} \binom{s - j + k - 3}{k - 3}.
$$
The $H_k^\pm(-s)$ Approximate $Z_f(s)$

**Theorem 3 (Ono-R-Sprung)**

Suppose that $k \geq 4$ and $\epsilon \in \{\pm1\}$. Then we have that

$$\lim_{N \to +\infty} Z_f(s) = H_k^\epsilon(-s),$$
The $H_k^\pm(-s)$ Approximate $Z_f(s)$

**Theorem 3 (Ono-R-Sprung)**

*Suppose that $k \geq 4$ and $\epsilon \in \{\pm 1\}$. Then we have that*

$$\lim_{N \to +\infty} Z_f(s) = H_k^\epsilon(-s),$$

*where $f \in S_k(\Gamma_0(N))$ are chosen with $\epsilon(f) = \epsilon$.***
The $H_k^\pm(-s)$ Approximate $Z_f(s)$

Theorem 3 (Ono-R-Sprung)

Suppose that $k \geq 4$ and $\epsilon \in \{\pm 1\}$. Then we have that

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where $f \in S_k(\Gamma_0(N))$ are chosen with $\epsilon(f) = \epsilon$.

Remark

This offers an unexpected connection to polytopes.
Ehrhart Polynomials

**Definition**

Given a \(d\)-dimensional integral lattice polytope in \(\mathbb{R}^n\), the **Ehrhart polynomial** \(\mathcal{L}_p(x)\) is determined by

\[
\mathcal{L}_p(m) = \# \{ p \in \mathbb{Z}^n : p \in mP \}.
\]
Ehrhart Polynomials

**Definition**

Given a $d$-dimensional integral lattice polytope in $\mathbb{R}^n$, the **Ehrhart polynomial** $\mathcal{L}_p(x)$ is determined by

$$\mathcal{L}_p(m) = \# \{ p \in \mathbb{Z}^n : p \in m\mathcal{P} \} .$$

**Example**

The polynomials $H_k^{-}(s)$ are the Ehrhart polynomials of the simplex

$$\text{conv} \left\{ e_1, e_2, \ldots, e_{k-3}, - \sum_{j=1}^{k-3} e_j \right\} .$$
Limits of $f \in S_6(\Gamma_0(N))$ with $\epsilon(f) = -1$

Figure: The tetrahedron whose Ehrhart polynomial is $H_6^-(s)$. 
Limits of $f \in S_6(\Gamma_0(N))$ with $\epsilon(f) = -1$

Figure: The tetrahedron whose Ehrhart polynomial is $H_6^-(s)$. 

$$\lim_{N \to +\infty} Z_f(s) = H_6^-(s) = -\frac{2}{3} \left( s - \frac{1}{2} \right) \left( s - \frac{1}{2} + \frac{\sqrt{-11}}{2} \right) \left( s - \frac{1}{2} - \frac{\sqrt{-11}}{2} \right).$$
Theorem 1 (Ono-R-Sprung)

*If* $f \in S_k(\Gamma_0(N))$ *is an even weight* $k \geq 4$ *newform, then we have:*

1. For all $s \in \mathbb{C}$ we have that $Z(f)(s) = \varepsilon(f)Z(f)(1-s)$.
2. If $Z(f)(\tau) = 0$, then $\Re(\tau) = 1/2$.

Theorem 2 (Ono-R-Sprung)

Define the period polynomial for $f$ by $R_f(z) := k/2 \sum_{j=0}^{\infty} \sqrt[k]{2j} \cdot \langle f, k_1^{1+j} \rangle \cdot z^j$.

Then we have that $R_f(z)(1-z)^k = \sum_{n=0}^{\infty} Z(f)(n)z^n$. 


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Proof of Theorems 1 and 2
Theorem 1 (Ono-R-Sprung)

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Theorem 2 (Ono-R-Sprung)

Define the **period polynomial** for \( f \) by

\[
R_f(z) := \sum_{j=0}^{k-2} \binom{k-2}{j} \cdot \Lambda(f, k-1-j) \cdot z^j.
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\]

Then we have that

\[
\frac{R_f(z)}{(1 - z)^{k-1}} = \sum_{n=0}^{\infty} Z_f(-n)z^n.
\]
Theorem (Rodriguez-Villegas (2002))

Suppose that $U(z) \in \mathbb{R}[z]$ is a degree $e$ polynomial with $U(1) \neq 0$. Then there is a polynomial $H(z)$ for which

$$\frac{U(z)}{(1 - z)^{e+1}} = \sum_{n=0}^{\infty} H(n)z^n.$$
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If all roots of $U(z)$ are on $|z| = 1$, then we have:
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If all roots of $U(z)$ are on $|z| = 1$, then we have:

1. All roots of $Z(s) := H(-s)$ lie on $\text{Re}(z) = 1/2$. 
Theorem (Rodriguez-Villegas (2002))

Suppose that \( U(z) \in \mathbb{R}[z] \) is a degree \( e \) polynomial with \( U(1) \neq 0 \). Then there is a polynomial \( H(z) \) for which

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If all roots of \( U(z) \) are on \( |z| = 1 \), then we have:

1. All roots of \( Z(s) := H(-s) \) lie on \( \text{Re}(z) = 1/2 \).
2. We have that

\[
Z(1 - s) = \pm Z(s).
\]
Proof of Theorems 1 and 2

Sketch of the proof of Theorems 1 and 2.

For even weight $k \geq 4$ newforms $f$ we must prove that $R_f(\tau) = 0$ for $|z| = 1$.

Make the definition of $Z_f(s) := H(s)\explicitly$ (i.e., Stirling numbers and weight moments).
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Sketch of the proof of Theorems 1 and 2.

- For even weight $k \geq 4$ newforms $f$ we **must prove** that
  \[ R_f(\rho) = 0 \implies |z| = 1. \]

- Make the definition of $Z_f(s) := H(-s)$ explicit (i.e. Stirling numbers and weight moments).
Definition
If \( f \in S_k(\mathbb{Z}) \) is a newform, then its period polynomial is
\[
r_f(X) := \sum_{m=1}^{k-1} L(f, k-1, m) \cdot (2\pi i X)^m.
\]

Problems (Open)
1. Determine the \( r_f(X) \).
2. Study the "distribution" of the zeros of \( r_f(X) \).
Zeta-polynomials for modular form periods
Proof of Theorems 1 and 2

Generating Function for Critical Values

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If \( f \in S_k(\Gamma_0(N)) \) is a newform, then its **period polynomial** is

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Definition

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Generating Function for Critical Values

Definition

If \( f \in S_k(\Gamma_0(N)) \) is a newform, then its \textbf{period polynomial} is

\[
r_f(X) := \sum_{m=0}^{k-2} L(f, k - 1 - m) \cdot \frac{(2\pi iX)^m}{m!}.
\]

Problems (Open)

1. Determine the \( r_f(X) \).
2. Study the “distribution” of the zeros of \( r_f(X) \).
Example. $f \in S_4(\Gamma_0(8))$

Let $f(\tau) = q - 4q^3 - 2q^5 + \cdots \in S_4(\Gamma_0(8))$ be the unique newform.
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\[
L(f, 1) \approx 0.354500683730965, \\
L(f, 2) \approx 0.690031163123398, \\
L(f, 3) \approx 0.874695377085079. 
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$$r_f(X) \approx -6.9975X^2 + 4.33559iX + 0.87469.$$  

3. Its roots are $\pm 0.170376720591406 + 0.309793113352311i$, which have norm$^2$ approximately $0.125000000 \approx \frac{1}{8}$.  

Suppose that \( f \in \mathcal{S}_k(\mathbb{N}) \) is a newform with \( k < 4 \).

If \( rf(z) = 0 \), then \( |z| = \frac{1}{p\mathbb{N}} \).

Remark: The circle \( |z| = \frac{1}{p\mathbb{N}} \) is the "symmetry" for a functional equation.
Conjecture (RHPP)

Suppose that \( f \in S_k(\Gamma_0(N)) \) is a newform with \( k \geq 4 \).
Zeta-polynomials for modular form periods
Proof of Theorems 1 and 2

“Riemann Hypothesis” for Period Polynomials

**Conjecture (RHPP)**

*Suppose that $f \in S_k(\Gamma_0(N))$ is a newform with $k \geq 4$. If $r_f(z) = 0$, then $|z| = \frac{1}{\sqrt{N}}$.***


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- In 2013 Conrey, Farmer, and Immamoğlu proved that zeros of the “odd part” of $r_f(X)$ have $|z| = 1$ when $N = 1$. 
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- El-Guindy and Raji proved the $N = 1$ case.
Recent results on RHPP

Theorem 4 (Jin-Ma-Ono-Soundararajan)
The Riemann Hypothesis for period polynomials is true.

Corollary (Jin-Ma-Ono-Soundararajan)
If $f \in S_k(\mathbb{N})$ is an even weight $k \geq 4$ newform, then all of the zeros of $R_f(z)$ satisfy $|z| = 1$.

In particular, Theorems 1 and 2 are true.
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Equidistribution

Theorem 5 (Jin-Ma-Ono-Soundararajan)

For fixed $\Gamma_0(N)$, as $k \to +\infty$, the zeros of $r_f(X) = 0$ become equidistributed on the circle with radius $\frac{1}{\sqrt{N}}$. 
Equidistribution

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**Question**

*Can one do better than equidistribution?*
Theorem 6 (Jin-Ma-Ono-Soundararajan)

If either $N$ or $k$ is large enough, then the roots of $r_f(X)$ are:

$$X \sim 1 \pm i\pi N \cdot \exp(i\pi/H) + O(1/k^2N),$$

where for $0 \leq \gamma \leq k$ we let $\gamma \in [0, 2\pi)$ be the solution to:

$$k \cdot \sin(\gamma N \cdot \pi) = \begin{cases} \pi, & \text{if } \varepsilon(f) = 1, \\ \gamma \pi, & \text{if } \varepsilon(f) = -1. \end{cases}$$

Remarks

For fixed $k$, the roots of $r_f(X)$ converge as $N \to +\infty$. This proves Theorem 3 that for fixed $\varepsilon(f)$ we have

$$\lim_{N \to +\infty} \mathcal{Z}_f(s) = H^{\pm k}(s).$$
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If either $N$ or $k$ is large enough, then the roots of $r_f(X)$ are:

$$X_\ell = \frac{1}{i \sqrt{N}} \cdot \exp \left( i \theta_\ell + O \left( \frac{1}{2^k \sqrt{N}} \right) \right),$$
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where for $0 \leq \ell \leq k - 3$ we let $\theta_\ell \in [0, 2\pi)$ be the solution to:

$$\frac{k - 2}{2} \cdot \theta_\ell - \frac{2\pi}{\sqrt{N}} \sin(\theta_\ell) = \begin{cases} 
\frac{\pi}{2} + \ell\pi & \text{if } \varepsilon(f) = 1, \\
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**Remarks**

- *For fixed* \(k\), *the roots of* \(r_f(X)\) *converge as* \(N \to +\infty\).
- *This proves Theorem 3 that for fixed* \(\epsilon(f) \in \{\pm\}\) *we have*

\[
\lim_{N \to +\infty} Z_f(s) = H_k^\pm(-s).
\]
Proof of RHPP when $k = 4$

- We care about the zeros of

$$-2L(f, 1)\pi^2 X^2 + 2\pi i L(f, 2)X + L(f, 3) = 0.$$
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- Therefore, we can use the quadratic equation.

- Using the functional equation to relate $L(f, 1)$ and $L(f, 3)$. The conclusion is trivial if $L(f, 2) = 0.$
Proof of RHPP when $k = 4$ cont.
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- If $L(f, 2) \neq 0$, then we need to show $\frac{N}{\pi^2} L(f, 3)^2 \geq L(f, 2)^2$. 
Proof of RHPP when $k = 4$ cont.

- If $L(f, 2) \neq 0$, then we need to show $\frac{N}{\pi^2} L(f, 3)^2 \geq L(f, 2)^2$.

- Then use Hadamard factorization of $\Lambda(f, s)$

$$
\Lambda(f, s) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \exp(s/\rho).
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- This means that $\Lambda(f, 3) \geq \Lambda(f, 2)$.
Lemma

If \( f \in S_k(\Gamma_0(N)) \) is a newform, then

\[
rf(X) = -\frac{(2\pi i)^{k-1}}{(k-2)!} \cdot \int_0^{i\infty} f(\tau) (\tau - X)^{k-2} d\tau.
\]
PSL$_2(\mathbb{R})^+$ action

**Definition**

If $\phi(z) \in \mathbb{C}[z]$ with $\deg(\phi) \leq k - 2$ and $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \text{PSL}_2(\mathbb{R})^+$, then

$$\phi \mid (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})(z) := (ad - bc)^{1 - \frac{k}{2}} \cdot (cz + d)^{k - 2} \cdot \phi \left( \frac{az + b}{cz + d} \right).$$
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**Remark**
This defines a “modular action” on

$$V_{k-2} := \{ \phi \in \mathbb{C}[z] : \deg(\phi) \leq k - 2 \}.$$
Lemma

If \( f \) is a newform, then \( p_f(X) := r_f(X/i) \in \mathbb{R}[X] \) satisfies:
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If $f$ is a newform, then $p_f(X) := r_f(X/i) \in \mathbb{R}[X]$ satisfies:

$$p_f(X) = \pm i^k \left( \sqrt{NX} \right)^{k-2} \cdot p_f \left( \frac{1}{NX} \right).$$
Functional Equation for $r_f(X)$

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**Proof.**

- If $\mathcal{W}_N := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$, then Atkin-Lehner implies

  $$f|\mathcal{W}_N = \pm f.$$
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- Since \( W_N^2 = I \) in \( \text{PSL}_2(\mathbb{R})^+ \),
Functional Equation for $r_f(X)$

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**Proof.**

- If $W_N := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$, then Atkin-Lehner implies

  $$f|W_N = \pm f.$$  

- Since $W_N^2 = I$ in $\text{PSL}_2(\mathbb{R})^+$, we get

  $$r_f | (1 \pm W_N) = 0.$$
General Strategy

1. Let \( m := \frac{k-2}{2} \), and define

\[
P_f(X) := \frac{1}{2} \binom{2m}{m} \Lambda \left( f, \frac{k}{2} \right) + \sum_{j=1}^{m} \binom{2m}{m+j} \Lambda \left( f, \frac{k}{2} + j \right) X^j.
\]
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Theorem 4 follows if the unit circle has all of the zeros of

$$T_f(X) := P_f(X) + \varepsilon(f) P_f(1/X).$$
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Letting \( X \to z = e^{i\theta} \) on \( |z| = 1 \), then \( T_f(z) \) is a “trigonometric” polynomial in sin or cos depending \( \epsilon(f) \).
Zeta-polynomials for modular form periods
Proving RHPP

Classical Theorem of Pólya and Szegö

Suppose that

\[ u(\xi) = a_0 + a_1 \cos(\xi) + a_2 \cos(2\xi) + \cdots + a_n \cos(n\xi), \]

\[ v(\xi) = a_1 \sin(\xi) + a_2 \sin(2\xi) + \cdots + a_n \sin(n\xi). \]

If

\[ 0 \leq a_0 \leq a_1 \leq a_2 \cdots \leq a_n, \]

then both \( u \) and \( v \) have exactly \( n \) zeros in \([0, \pi]\), and these zeros are simple.
Classical Theorem of Pólya and Szegő

Theorem (Szegö, 1936)

Suppose that $u(\theta)$ and $v(\theta)$ are

$$u(\theta) := a_0 + a_1 \cos(\theta) + a_2 \cos(2\theta) + \cdots + a_n \cos(n\theta),$$

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    v(\theta) := a_1 \sin(\theta) + a_2 \sin(2\theta) + \cdots + a_n \sin(n\theta).
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If $0 \leq a_0 \leq a_1 \leq a_2 \cdots \leq a_{n-1} < a_n$, then both $u$ and $v$ have exactly $n$ zeros in $[0, \pi)$, and these zeros are simple.
Lemma 1

The completed $L$-function $\Lambda(f, s)$ satisfies the following:

1) It is **monotone increasing** in the range $s \geq \frac{k}{2} + \frac{1}{2}$. 
Useful inequalities

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2) In particular, we have

$$0 \leq \Lambda\left(f, \frac{k}{2}\right) \leq \Lambda\left(f, \frac{k}{2} + 1\right) \leq \Lambda\left(f, \frac{k}{2} + 2\right) \leq \ldots.$$
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3) If $\epsilon(f) = -1$, then $\Lambda \left( f, \frac{k}{2} \right) = 0$ and

$$\Lambda \left( f, \frac{k}{2} + 1 \right) \leq \frac{1}{2} \Lambda \left( f, \frac{k}{2} + 2 \right) \leq \frac{1}{3} \Lambda \left( f, \frac{k}{2} + 3 \right) \leq \ldots.$$
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- Check remaining cases using SAGE.
Our results
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*Manin’s Conjecture is true.*
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1. For each newform \( f \) there is a zeta-polynomial \( Z_f(s) \) which has a FE and obeys RH.
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**Theorem (Ono-R-Sprung)**

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2. The \( Z_f(-n) \) encode the “Bloch-Kato complex.”
Theorem (Ono-R-Sprung)

*Manin’s Conjecture is true.*

1. For each newform $f$ there is a zeta-polynomial $Z_f(s)$ which has a FE and obeys RH.
2. The $Z_f(-n)$ encode the “Bloch-Kato complex.”
3. For fixed $k$ and $\varepsilon(f) = \varepsilon$, we have

$$\lim_{N \to +\infty} Z_f(s) = H_k^\varepsilon(-s).$$
This makes use of the following new result.

**Theorem 4 (Jin-Ma-Ono-Soundararajan)**

*The Riemann Hypothesis for period polynomials is true.*