DETERMINANTS

LINEAR ALGEBRA

1. Determinants

We have just seen some applications of invertibility of matrices. We would now like to describe how to detect whether a matrix is invertible. We recently proved that if a matrix is invertible, then its RREF is I_n. In class, we saw that if $ad - bc \neq 0$, then the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ reduces to I_2 and is hence invertible. Following the steps of that proof, it isn't hard to see that in fact this is an if and only if statement.

It will turn out for every square matrix of any size that there is a number associated to it, called the **determinant**, which vanishes if and only if the matrix isn't invertible. In the case of 2×2 matrices, this happens to be the number $ad - bc$. For general n, there is a unique function $\det A$ of matrices of size $n \times n$ satisfying a few simple properties. We will first think of the determinant as a function of n variables, $\det(r_1, \ldots, r_n)$, where r_i is the i-th row of A, thought of as a vector. The characterizing properties of det are the following, which give a it definition of it, as we'll see:

(1) det is a multilinear function.
(2) det is **alternating**, which means that if any two rows of A are equal, say $r_i = r_j$, then the determinant is 0.
(3) The value of det on I_n is 1.

In this definition, a *multilinear function* means one which takes in inputs a bunch of vectors from different vector spaces and spits out a vector and which is linear in each input variable. For example, it could take a tuple $(v_1, v_2, \ldots v_k)$ of vectors $v_i \in V_i$ for vector spaces $V_1, \ldots, V_k$, and spit out a scalar, and multilinearity would mean that it is linear in each component simultaneously.

Here is a simple consequence of the above properties: if we swap two rows in A, say rows i and j, then we have, by the alternating property and by multilinearity,

$$\begin{aligned} 0 &= \det(r_1, \ldots, r_i + r_j, \ldots, r_i + r_j, \ldots, r_n) \\ &= \det(r_1, \ldots, r_i, \ldots, r_i, \ldots, r_n) + \det(r_1, \ldots, r_i, \ldots, r_j, \ldots, r_n) \\ &+ \det(r_1, \ldots, r_j, \ldots, r_i, \ldots, r_n) + \det(r_1, \ldots, r_j, \ldots, r_j, \ldots, r_n) \\ &= \det(r_1, \ldots, r_i, \ldots, r_j, \ldots, r_n) + \det(r_1, \ldots, r_j, \ldots, r_i, \ldots, r_n), \end{aligned}$$

so that **swapping two rows of a matrix multiplies the determinant by -1.** Moreover, if we add a multiple c of row j to row i, we find, again using multilinearity and the alternating property, that

$$\begin{aligned}&\det(r_1, \ldots, r_i + cr_j, \ldots, r_j, \ldots, r_n)\\&= \det(r_1, \ldots, r_i, \ldots, r_j, \ldots, r_n) + c\det(r_1, \ldots, r_j, \ldots, r_j, \ldots, r_n)\\&= \det(r_1, \ldots, r_i, \ldots, r_j, \ldots, r_n),\end{aligned}$$

so that **adding a multiple of one row to another leaves a determinant unchanged.** Finally, if we multiply a row by a constant, then multilinearity again shows that **the determinant is multiplied by the same constant.**

Example. *We use the properties above to find that (with* $e_1 = (1, 0)$ *and* $e_2 = (0, 1)$*)*

$$\begin{aligned}&\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det((a, b), (c, d)) = \det(ae_1 + be_2, ce_1 + de_2)\\&ac\det(e_1, e_1) + ad\det(e_1, e_2) + bc\det(e_2, e_1) + bd\det(e_2, e_2)\\&= ad\det(e_1, e_2) - bc\det(e_1, e_2) = ad\det I_2 - bc\det I_2 = ad - bc.\end{aligned}$$

Example. *We compute (with* $e_1 = (1, 0, 0)$*,* $e_2 = (0, 1, 0)$*, and* $e_3 = (0, 0, 1)$*)*

$$\begin{aligned}&\det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} = \det(e_1 + 2e_2 + 3e_3, e_1 + 2e_3, e_2)\\&= \det(e_1, e_1 + 2e_3, e_2) + 2\det(e_2, e_1 + 2e_3, e_2) + 3\det(e_3, e_1 + 2e_3, e_2)\\&= \det(e_1, e_1, e_2) + 2\det(e_1, e_3, e_2) + 2\det(e_2, e_1, e_2) + 4\det(e_2, e_3, e_2) + 3\det(e_3, e_1, e_2) + 6\det(e_3, e_3, e_2)\\&= 2\det(e_1, e_3, e_2) + 3\det(e_3, e_1, e_2) = -2\det(e_1, e_2, e_3) + 3\det(e_1, e_2, e_3) = -2\det I_3 + 3\det I_3 = 1.\end{aligned}$$

After doing a few such numerical examples, you discover that there are some patterns which seem to emerge. We will explain this by giving another definition of the determinant. Firstly, however, we need to describe a new mathematical object.

2. Permutations

A **permutation** of a set with n elements is simply a rearrangement of the elements. We will usually describe these by taking as a set of size n the set of the first n natural numbers $1, 2, \ldots, n$. For example, we may rearrange the numbers $1, 2, 3$ according to a permutation π, giving $3, 1, 2$. One convenient way of keeping track of this action is to use the two row notation:

$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

In this notation, the numbers in the second row are the results of applying the permutation to the elements of the first row. There is one very special type of permutation we shall need, called a **transpostion**. We will denote by (ij) the permutation which only switches i and j and leaves all other elements unchanged. Note that we must know by context what n is, since (ij) can denote a permutation of any number n of elements. For example,

$$(23) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

More generally, a k-**cycle** is a permutation $(a_1, a_2, \ldots, a_k)$ which sends a_1 to a_2, a_2 to $a_3, \ldots$ a_{k-1} to a_k and finally a_k to a_1, and leaves all elements not listed fixed. For example, we have

$$(152) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 3 & 4 & 2 \end{pmatrix}.$$

We can define products of permutations by writing them next to one another and by applying the actions of each working from right to left (the reason being that these are really compositions of functions). This will give a new permutation.

Example. *If we consider two permutations*

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix},$$

then

$$\pi\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 4 & 2 \end{pmatrix}, \qquad \sigma\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 & 1 \end{pmatrix}.$$

Note that multiplication of permutations is **not** *commutative.*

What we are interested in is decomposing permutations into cycles. This can be done using the following simple procedure.

Algorithm. *To write a permutation π as a product of disjoint cycles (i.e., a product of cycles with no common elements between any two of them), pick the first number among $1, 2, \ldots n$ which isn't fixed (going to itself) by π. This is the first element of the first cycle. To find the rest of the first cycle, keep applying π to successive elements until you get back to the first element you started with, in which case you have "closed off" the first cycle. Now repeat this process on the set of all remaining numbers from $1, \ldots n$ until every element is either fixed by π or is in one of the cycles you have already written down.*

Example. *For the permutations in the last example, we have*

$$\pi = (1243),$$

and

$$\sigma = (15)(24).$$

We also have

$$\pi\sigma = (1523),$$

and

$$\sigma\pi = (1435).$$

The key application of these cycle decompositions is the following result, whose proof would require too much time for the application we have in mind in this class.

Theorem. *If π is is any permutation, then there is a certain unique number ± 1, called* $\text{sign}(\pi)$ *or the sign of π, associated to π. If π can be written as a product of m transpositions (this in general isn't unique, but the claim is that its true for* **any** *representation as a product of transpositions that you find), then* $\text{sign}(\pi) = (-1)^m$.

We also say that π is **even** if $\text{sign}(\pi) = +1$ and that π is **odd** if $\text{sign}(\pi) = -1$. The point is that every permutation can be written as a product of transpositions. This can be found by first finding the cycle decomposition of the preceding algorithm and then using the following elementary decomposition of any cycle into transpositions:

$$(a_1, \dots a_k) = (a_1 a_k) \cdots (a_1 a_3)(a_1 a_2).$$

This directly shows that any k cycle has sign $(-1)^{k+1}$.

Note that the parity of an k-cycle as a permutation is **opposite** the parity of k as an integer.

Thus, if we use the algorithm above, then the parity of any permutation which is a product of cycles of lengths $k_1, \dots, k_\ell$ can be read off as the product $(-1)^{k_1+1} \cdots (-1)^{k_\ell+1}$.

Example. *Assuming the notation of the last example, we can use the cycle decompositions directly to read off the signs of all the permutations involved:*

$$\text{sign}(\pi) = (-1)^{4+1} = -1,$$

and

$$\text{sign}(\sigma) = (-1)^3(-1)^3 = +1,$$
$$\text{sign}(\pi\sigma) = (-1)^5 = -1,$$

and

$$\text{sign}(\sigma\pi) = (-1)^5 = -1.$$

3. Leibniz form of the determinant

Using the reasoning in the above examples for determinants, we can write down a general formula for determinants using permutations. As we extrapolate from the examples above, we can see that if we want to use the defining properties of a determinant to compute it, we first write each row vector in terms of the **standard basis vectors** $e_1, \dots, e_n$, where e_i is the i-th row of I_n, and then use multilinearity to expand. We will then get a sum of products of coordinates of the row vectors, namely matrix entries, with one term in each product coming from each row, times values of determinants on some ordering of the e_i's. Now, by the alternating property, whenever one of these e_j functions appears twice, we will get a zero in that term. Otherwise, we are exactly in the case that the entries in the corresponding term are just a permutation of $e_1, \dots, e_n$.

Finally, using the property that switching two rows just multiplies the determinant by a factor of -1, we take each of these remaining terms and perform a series of transpositions (swapping two entries), to reduce it to the sign of the corresponding permutation times the determinant of I_n, which is of course 1. All of this is summarized in the following result, where if $x = 1, \dots, n$ and π is a permutation, then $\pi(x)$ is the result of applying π to x.

Theorem. *The determinant function defined by the properties above can be computed for any matrix A as*

$$\det A = \sum_{\pi} \operatorname{sign}(\pi) A_{1\pi(1)} A_{2\pi(2)} \cdots A_{n\pi(n)},$$

where in the sum, π runs over all permutations of $1, 2, \dots n$.

4. Determinants as a criterion for detecting invertibility

We next show a few very important properties of determinants.

Theorem. *The following hold for any $n \times n$ matrices A, B.*

1. *If A has a row of zeros, then* $\det A = 0$.
2. $\det(AB) = \det A \cdot \det B$.
3. *A is invertible if and only if* $\det A \neq 0$.
4. *If A is invertible, then* $\det(A^{-1}) = (\det A)^{-1}$.

Proof. (1): If the i-th row r_i of A is zero, then $cr_i = r_i$ for any c, so that for a fixed $c \neq 0$, using mulilinearity of the determinant as a function on the rows of A, we have

$$\det A = \det(r_1, \dots, r_i, \dots, r_n) = \det(r_1, \dots, cr_i, \dots, r_n) = c \det(r_1, \dots, r_i, \dots, r_n) = c \det A,$$

which implies $\det A = 0$.

(2):

We first show it when $A = E$ is an elementary matrix. By the properties of det we showed before, $\det EB = \det B, -\det B, c \det B$, depending on whether E corresponds to adding a multiple of one row to another row, switching two rows, or multiplying a row by a constant, respectively. It is also obvious that $\det E = 1, -1, c$ accordingly, as E is obtained by performing the same elementary row operation on I_n, which has determinant 1.

In general, there are two cases. If A is invertible, by our previous theorem, A is a product of elementary matrices, so by applying the argument in the last paragraph repeatedly, the result follows. On the other hand, if A isn't invertible, then the RREF of A isn't I_n, meaning that there isn't a pivot in some row, meaning that A has a row of zeros. This means that there are elementary matrices E_j for which $E_1 \cdots E_k A$ has a row of zeros, and by the argument in the last paragraph and by part (1), the product $\det(E_1) \cdots \det(E_k) \det(A) = 0$. As the determinant of an elementary matrix is never zero, we find that $\det A = 0$. We are done if we can show $\det AB = 0$. This follows since $E_1 \cdots E_k A$ has a row of zeros, and so $E_1 \cdots E_k AB$ has a row of zeros as well.

(3): If A is invertible, then it is a product of elementary matrices, and so by (2) has non-zero determinant. Conversely, if A isn't invertible, then we saw in the proof of (2) that $\det A = 0$.

(4): By definition, $I_n = AA^{-1}$, and so by (2), we have $\det(I_n) = 1 = \det A \det A^{-1}$, which implies the claim.

□

5. Matrix transposes

Given any matrix A of size $m \times n$, there is a matrix A^T, called the **transpose** of A, which has size $n \times m$. This is obtained by reflecting A across its main diagonal. Another way of thinking is that the rows of one are the columns of the other. Formally, we have the following.

Definition. For any matrix A of size $m \times n$, the transpose of A, written A^T, is the $n \times m$ matrix with

$$(A^T)_{ji} = A_{ij}.$$

Example. *If*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix},$$

then

$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

Example. *For any matrices A, B of the same size, and for any constant c, we have*

$$(A + B)^T = A^T + B^T,$$

$$(cA)^T = c(A^T),$$

and

$$(A^T)^T = A.$$

Example. *If $a, b \in \mathbb{R}^n$ are vectors, thought of as columns, then the dot product may be expressed*

$$a \cdot b = a^T b.$$

Transposes satisfy a property not unlike the socks and shoes property for inverses.

Lemma. *For any $m \times n$ matrix A and any $n \times \ell$ matrix B, we have $(AB)^T = B^T A^T$. More generally, we have (whenever both sides make sense):*

$$(A_1 \ldots A_k)^T = A_k^T \ldots A_1^T.$$

Proof. It is enough to show the first claim, as the general claim follows by repeatedly applying the general claim with $k = 2$. This is just an explicit calculation:

$$((AB)^T)_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk}B_{ki} = \sum_{k=1}^{n} B_{ki}A_{jk} = \sum_{k=1}^{n} (B^T)_{ik}(A^T)_{kj} = (B^TA^T)_{ij}.$$

□

Transposes also play nicely with determinants.

Lemma. *For any $n \times n$ matrix A,*

$$\det(A^T) = \det A.$$

Proof. There are two cases. If A is invertible, then A is a product $A = E_1 \cdots E_k$ of elementary matrices. Thus, $A^T = E_k^T \cdots E_1^T$. As a determinant of a product is the product of determinants, it is enough to show that $\det E^T = \det E$ for any elementary matrix. Indeed, if E switches two rows, or if E multiplies a row by a constant, then $E = E^T$, so their determinants are clearly equal. If E adds a multiple of one row to another, then $\det E = 1$, and E^T is another elementary matrix of the same type, so $\det(E^T) = 1$ as well.

Now, if A isn't invertible. then A^T isn't either, for if it was, then $A^TB = I_n$ implies $(A^TB)^T = B^T(A^T)^T = B^TA = I_n$, which implies that A is invertible, which is a contradiction. (If you want to be precise for the moment and check the equation $BA^T = I_n$ as well, you can, yielding that $B^TA = AB^T = I_n$, which was our original definition of invertible). Thus, the determinants of both A and A^T are zero. □

This result is very handy in many computations, as it allows us to think of columns instead of rows, which may be more convenient for explicit examples. In particular, we have the following corollary.

Theorem. *The determinant is also a multilinear, alternating function of the columns of a matrix.*

In particular, any properties you used regarding elementary row operations, hold true in exactly the same way if we replace the word "row" everywhere with "column". For example, switching two columns of a matrix multiplies the determinant by -1.

6. Minors and cofactors

Our definition of determinants is really, really, tedious to check for large matrices. The original definition requires one to evaluate n^n terms, while the Leibniz formula, which got rid of lots of terms by the alternating property, still requires one to evaluate $n!$ terms. This still grows exponentially with n. However, the determinant can be evaluated in polynomial time. So, the definitions we have given are ideal for proving theorems, but not ideal for computations. In general, somewhat like integration, finding determinants of large matrices efficiently, or finding closed formulas for determinants of infinite families of matrices is a bit of an art form. Many techniques exist, including one very handy one by Charles Dodgson (aka Lewis Carroll, of Alice in Wonderland fame).

We will now describe one method which is very widely useful, and so you never have to enumerate permutations to compute determinants again.

Definition. Given an $n \times n$ matrix A, the (i, j)-th **minor**, denoted A^{ij}, is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i-th row and the j-th column. Similarly, the (i, j)-th **cofactor** C^{ij} is defined in terms of the minor by

$$C^{ij} = (-1)^{i+j} A^{ij}.$$

Example. *For*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix},$$

we have

$$A^{23} = \det \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix} = 8 - 14 = -6.$$

We also find $C^{23} = (-1)^{2+3}(-6) = 6$.

Next, we will see how these minors give a very simple, easily evaluated expression for determinants of matrices.

7. Laplace expansions

By using the cofactors from the last lecture, we can find a very convenient way to compute determinants. We first give the method, then try several examples, and then discuss its proof.

Algorithm (Laplace expansion)**.** *To compute the determinant of a square matrix, do the following.*

1. *Choose any row or column of* A*.*
2. *For each element* A_{ij} *of this row or column, compute the associated cofactor* C^{ij}*.*
3. *Multiply each cofactor by the associated matrix entry* A_{ij}*.*
4. *The sum of these products is* $\det A$*.*

Example. *We find the determinant of*

$$A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{pmatrix}.$$

We make the arbitrary choice to expand along the first row. We compute the minors as

$$M^{11} = \det \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}, \qquad M^{12} = \det \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix}, \qquad M^{13} = \det \begin{pmatrix} -1 & 2 \\ -2 & 2 \end{pmatrix}.$$

Computing these 2×2 *determinants, we have*

$$M^{11} = 4, \qquad M^{12} = -1, \qquad M^{13} = 2.$$

By inserting signs, we find that the cofactors are

$$C^{11} = M^{11} = 4, \qquad C^{12} = -M^{12} = 1, \qquad C^{13} = M^{13} = 2.$$

Thus,

$$\det A = A_{11}C^{11} + A_{12}C^{12} + A_{13}C^{13} = 2(4) + (1) + 3(2) = 15.$$

With this method, much larger determinants are also feasible, especially when there are lots of zeros in a row or column. Generally speaking, it is a good idea when computing an example to try to expand along a row or column with as many zeros as possible, or at least with the smallest entries possible.

Example. *We find the determinant of*

$$A = \begin{pmatrix} 2 & 5 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 0 \\ -1 & 6 & 4 & 0 \end{pmatrix}.$$

We expand along the last column to find

$$\det A = 2 \det \begin{pmatrix} -2 & -3 & 2 \\ 1 & 3 & -2 \\ -1 & 6 & 4 \end{pmatrix} - 5 \det \begin{pmatrix} 2 & 5 & -3 \\ 1 & 3 & -2 \\ -1 & 6 & 4 \end{pmatrix}$$

We can find these two determinants by expanding them as well. For example, expanding along the first column on the first one, we find that

$$\det \begin{pmatrix} -2 & -3 & 2 \\ 1 & 3 & -2 \\ -1 & 6 & 4 \end{pmatrix} = -2 \det \begin{pmatrix} 3 & -2 \\ 6 & 4 \end{pmatrix} - \det \begin{pmatrix} -3 & 2 \\ 6 & 4 \end{pmatrix} - \det \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}$$
$$= -2(24) - (-24) - 0 = -48 + 24 + 0 = -24.$$

Similarly, by expanding the second 3×3 *matrix along the first column, we find that*

$$\begin{pmatrix} 2 & 5 & -3 \\ 1 & 3 & -2 \\ -1 & 6 & 4 \end{pmatrix} = 2 \det \begin{pmatrix} 3 & -2 \\ 6 & 4 \end{pmatrix} - \det \begin{pmatrix} 5 & -3 \\ 6 & 4 \end{pmatrix} - \det \begin{pmatrix} 5 & -3 \\ 3 & -2 \end{pmatrix}$$
$$= 2(24) - (38) - (-1) = 11.$$

Thus, we find that

$$\det A = 2(-24) - 5(11) = -103.$$

Discussion of the proof of the algorithm. Since we know that switching two rows negates determinants (and negates the sings in the cofactors as well), and since transposes preserve determinants (meaning that we can switch the roles of rows and columns in determinant calculations), it is enough to show this when we pick the first row in the algorithm. By our definition, it is enough to show that this satisfies the 3 properties uniquely characterizing determinants. That is, if we define $f(A) = A_{11}C^{11} + \ldots + A_{1n}C^{1n}$, then we just have to show that f is multilinear in the rows of A, that it is alternating in the rows, and that $f(I_n) = 1$. The proof of multilinearity, and of the alternating property, follow from careful writing down of the objects involved, but you can also try an example and are encouraged to think through why you should believe this! For example, you are strongly encouraged to just try out these two properties on an arbitrary, say 3×3, matrix. □

8. Adjugate matrices and inverses

In addition to finding determinants quickly, we can use cofactors to quickly compute inverses of matrices. If we stick all the cofactors into a matrix, then we obtain the cofactor matrix. That is, the cofactor matrix is the matrix C such that

$$C_{ij} = C^{ij}.$$

The **adjugate matrix** (sometimes called the adjoint matrix), denoted $\mathrm{adj}(A)$, is simply the transpose of the cofactor matrix:

$$(\mathrm{adj}A)_{ij} = C^{ji}.$$

The reason this matrix is interesting is that the following result holds.

Theorem. *For any $n \times n$ matrix A, we have*

$$A \cdot \mathrm{adj}(A) = \det(A) I_n.$$

In particular, if A is invertible, then $A^{-1} = (\det A)^{-1}\mathrm{adj}(A)$.

Proof. This is essentially a restatement of the Laplace expansion algorithm above. To check it, compute the i, j-th entry of the left hand side:

$$(A \cdot \mathrm{adj}A)_{ij} = \sum_{k=1}^{n} A_{ik}(\mathrm{adj}A)_{kj} = \sum_{k=1}^{n} a_{ik}C^{jk}.$$

If $i = j$, then by Laplace expansion, we get $\det A$. If $i \neq j$, then by Laplace expansion again, we are really computing the determinant of the matrix where we replace the j-th row of A by its i-th row. But such a matrix has two rows which are the same, and hence has determinant zero. □

Example. *We continue working with the matrix*

$$A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{pmatrix}$$

from above. We already found the first few cofactors to be

$$C^{11} = M^{11} = 4, \qquad C^{12} = -M^{12} = 1, \qquad C^{13} = M^{13} = 2.$$

Continuing in the same manner as above, we find that the matrix of cofactors is

$$\begin{pmatrix} 4 & 1 & 2 \\ 3 & 12 & -6 \\ -5 & -5 & 5 \end{pmatrix}.$$

Taking the transpose of this matrix yields that

$$\operatorname{adj}(A) = \begin{pmatrix} 4 & 3 & -5 \\ 1 & 12 & -5 \\ 2 & -6 & 5 \end{pmatrix}.$$

Since we saw above that $\det A = 15$*, we find that*

$$A^{-1} = \begin{pmatrix} \frac{4}{15} & \frac{1}{5} & -\frac{1}{3} \\ \frac{1}{15} & \frac{4}{5} & -\frac{1}{3} \\ \frac{2}{15} & -\frac{2}{5} & \frac{1}{3} \end{pmatrix}.$$

9. Cramer's Rule

Suppose that A is invertible. Then we already know that $Ax = b$ has only one solution for any b. Of course, this solution is the vector $x = A^{-1}b$. Plugging in the adjugate yields that

$$x_j = (\det A)^{-1}(\operatorname{adj} A)b = (\det A)^{-1} \sum_{k=1}^{n} C^{kj} b_k.$$

But the sum is just a j-th column expansion of the matrix A_j obtained by replacing the j-th column with b. This gives Cramer's fomula.

Theorem (Cramer's rule)**.** *Assume the notation above. If A is invertible, then the solution to $Ax = b$ is given by*

$$x_j = \frac{\det(A_j)}{\det A}.$$

Example. *We saw above that*

$$A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{pmatrix}$$

is invertible, with $\det A = 15$*. Thus, by Cramer's rule, the solution to* $Ax = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$ *is given by*

$$x_1 = \frac{1}{15} \det \begin{pmatrix} 1 & 1 & 3 \\ -2 & 2 & 1 \\ 0 & 2 & 3 \end{pmatrix} = -\frac{2}{15},$$

$$x_2 = \frac{1}{15} \det \begin{pmatrix} 2 & 1 & 3 \\ -1 & -2 & 1 \\ -2 & 0 & 3 \end{pmatrix} = -\frac{23}{15},$$

$$x_3 = \frac{1}{15} \det \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & -2 \\ -2 & 2 & 0 \end{pmatrix} = \frac{14}{15}.$$

10. 3×3 determinants

We have seen quite a lot about determinants, from their definitions to their applications to invertibility questions of matrices and in describing solutions of systems of equations, and concluding with some methods for computing them. It turns out that the Laplace expansion method is still pretty inefficient in general, but better methods (such as decomposing matrices into triangular matrices) are studied in a more advanced linear algebra course.

However, there are two cases when we can just evaluate determinants directly. The first is the case of 2×2 matrices, which we saw can be evaluated by the simple formula (which you should just memorize):

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

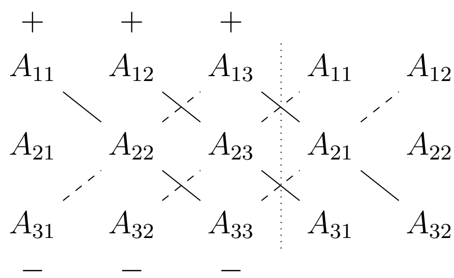
For 3×3 matrices, there is another trick, called the **rule of Sarrus**, which you are free to assume for the rest of the course. Note that this is **only applicable in the** 3×3 **case**; in general similar diagrams you could draw for finding determinants won't work. Without proof, we can state the method as follows. Consider the 3×3 matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

whose determinant we wish to compute, and copy over its left two columns and write them to the right of the matrix:

$$\begin{matrix} A_{11} & A_{12} & A_{13} & A_{11} & A_{12} \\ A_{21} & A_{22} & A_{23} & A_{21} & A_{22}. \\ A_{31} & A_{32} & A_{33} & A_{31} & A_{32} \end{matrix}$$

Now draw 6 diagonal slashes labelled with $\pm$ signs as follows:



The determinant of A will then be the sum of the six products of terms along these diagonals weighted by the corresponding signs. That is,

$$\det A = A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{31}A_{22}A_{13} - A_{32}A_{23}A_{11} - A_{33}A_{21}A_{12}.$$

11. Exercises

Here are a few sample exercises.

(1) Suppose that

$$A = \begin{pmatrix} 4 & 0 \\ 2 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \end{pmatrix}.$$

(a) Compute AB and BA.

(b) Compute, or show that it doesn't exist, the following inverses: $(AB)^{-1}$ and $(BA)^{-1}$.

Solution:

(a):

We find that

$$AB = \begin{pmatrix} 0 & 4 & 8 \\ 0 & 2 & 4 \\ 1 & 0 & -3 \end{pmatrix},$$

and

$$BA = \begin{pmatrix} 2 & -2 \\ -4 & -3 \end{pmatrix}.$$

(b):

The first matrix AB isn't invertible. For example, note that the first row is twice the second, and so we can row-reduce AB to have a row all zeros. Hence, its RREF isn't I_3, and it isn't invertible.

The second matrix is invertible. We compute it by row reducing the augmented matrix

$$\left(\begin{array}{cc|cc} 2 & -2 & 1 & 0 \\ -4 & -3 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & -1 & \frac{1}{2} & 0 \\ -4 & -3 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & -1 & \frac{1}{2} & 0 \\ 0 & -7 & 2 & 1 \end{array}\right)$$
$$\longrightarrow \left(\begin{array}{cc|cc} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{7} & -\frac{1}{7} \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & \frac{3}{14} & -\frac{1}{7} \\ 0 & 1 & -\frac{2}{7} & -\frac{1}{7} \end{array}\right),$$

and so $(BA)^{-1} = \left(\begin{smallmatrix} \frac{3}{14} & -\frac{1}{7} \\ -\frac{2}{7} & -\frac{1}{7} \end{smallmatrix}\right)$.

(2) For any 2×2 matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, define the trace of A, denoted $\operatorname{tr}(A)$, as the sum of diagonal elements, i.e. $\operatorname{tr}(A) = a + d$. Further let A^2 be the product $A \cdot A$ for any square matrix A. Show that for any 2×2 matrix, we have

$$2 \det A = \det \begin{pmatrix} \operatorname{tr}(A) & 1 \\ \operatorname{tr}(A^2) & \operatorname{tr}(A) \end{pmatrix}.$$

Hint: you may use the formula

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Solution:
We first compute

$$\operatorname{tr}(A) = a + d,$$
$$\operatorname{tr}(A^2) = \operatorname{tr}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \operatorname{tr} \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} = a^2 + 2bc + d^2.$$

Hence,

$$\det \begin{pmatrix} \operatorname{tr}(A) & 1 \\ \operatorname{tr}(A^2) & \operatorname{tr}(A) \end{pmatrix} = \det \begin{pmatrix} a + d & 1 \\ a^2 + 2bc + d^2 & a + d \end{pmatrix}$$
$$= (a+d)^2 - (a^2 + 2bc + d^2) = 2ad - 2bc = 2 \det A.$$

(3) Consider the permutations

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 7 & 4 & 1 & 5 & 2 & 6 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 1 & 7 & 5 & 6 \end{pmatrix}.$$

(a) Compute $\pi\sigma$.
(b) Write $\pi\sigma$ as a product of disjoint cycles.
(c) Use your answer from the last part to write $\pi\sigma$ as a product of transpositions.
(d) Use you answer from the last part to find $\operatorname{sign}(\pi\sigma)$.

Solution:
(a):
We find that

$$\pi\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 4 & 1 & 3 & 6 & 5 & 2 \end{pmatrix}.$$

(b): We compute $\pi\sigma = (17243)(56)$.

(c): We find that $\pi\sigma = (13)(14)(12)(17)(56)$.

(d): Since there were 5 transpositions in the representation for $\pi\sigma$ we found in (c), $\pi\sigma$ is an odd permutation, i.e., $\operatorname{sign}(\pi\sigma) = -1$.

(4) Compute $\det A$ **directly from the definition** (that is, as the unique alternating multilinear function on rows which has $\det I_n = 1$), where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 7 & 0 & 9 \end{pmatrix}.$$

Solution:

Writing this as a function of rows, and using multilinearity and the alternating property of det, we find that

$$\begin{aligned}
&\det A = \det(e_1 + 2e_2 + 3e_3, 5e_2, 7e_1 + 9e_3) \\
&= 5\det(e_1, e_2, 7e_1 + 9e_3) + 10\det(e_2, e_2, 7e_1 + 9e_3) + 15\det(e_3, e_2, 7e_1 + 9e_3) \\
&= 35\det(e_1, e_2, e_1) + 45\det(e_1, e_2, e_3) + 70\det(e_2, e_2, e_1) + 90\det(e_2, e_2, e_3) \\
&+ 105\det(e_3, e_2, e_1) + 135\det(e_3, e_2, e_3) \\
&= 45\det(e_1, e_2, e_3) + 105\det(e_3, e_2, e_1) = 45\det(e_1, e_2, e_3) - 105\det(e_1, e_2, e_3) \\
&= 45\det(I_3) - 105\det(I_3) = 45 - 105 = -60.
\end{aligned}$$

(5) Find the determinant of the matrix

$$A = \begin{pmatrix} 3 & 2 & 0 & 1 \\ 4 & 0 & 1 & 2 \\ 3 & 0 & 2 & 1 \\ 9 & 2 & 3 & 1 \end{pmatrix}.$$

Solution: We expand along the second column (since it has the most zeros):

$$\det A = -2\det\begin{pmatrix} 4 & 1 & 2 \\ 3 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix} + 2\det\begin{pmatrix} 3 & 0 & 1 \\ 4 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}.$$

In the first 3×3 matrix, we expand along the second row (as it has the smallest entries) to find that

$$\begin{aligned}
&\det\begin{pmatrix} 4 & 1 & 2 \\ 3 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix} = -3\det\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} + 2\det\begin{pmatrix} 4 & 2 \\ 9 & 1 \end{pmatrix} - \det\begin{pmatrix} 4 & 1 \\ 9 & 3 \end{pmatrix} \\
&= -3(-5) + 2(-14) - (3) = 15 - 28 - 3 = -16.
\end{aligned}$$

On the other 3×3 matrix, we expand along the second column to find:

$$\det \begin{pmatrix} 3 & 0 & 1 \\ 4 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} = \det \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} - 2 \det \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$$
$$= 0 - 2(2) = -4.$$

Overall, we find that $\det A = 24$.

(6) Use the method of adjoints to compute the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 7 & 0 & 9 \end{pmatrix}.$$

Solution: We first compute the minor matrix of A to be

$$\begin{pmatrix} \det \begin{pmatrix} 5 & 0 \\ 0 & 9 \end{pmatrix} & \det \begin{pmatrix} 0 & 0 \\ 7 & 9 \end{pmatrix} & \det \begin{pmatrix} 0 & 5 \\ 7 & 0 \end{pmatrix} \\ \det \begin{pmatrix} 2 & 3 \\ 0 & 9 \end{pmatrix} & \det \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix} & \det \begin{pmatrix} 1 & 2 \\ 7 & 0 \end{pmatrix} \\ \det \begin{pmatrix} 2 & 3 \\ 5 & 0 \end{pmatrix} & \det \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} & \det \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 45 & 0 & -35 \\ 18 & -12 & -14 \\ -15 & 0 & 5 \end{pmatrix}.$$

By inserting signs, we find the cofactor matrix to be

$$\begin{pmatrix} 45 & 0 & -35 \\ -18 & -12 & 14 \\ -15 & 0 & 5 \end{pmatrix}.$$

By taking the transpose, we find the adjoint matrix to be

$$\operatorname{adj}(A) = \begin{pmatrix} 45 & -18 & -15 \\ 0 & -12 & 0 \\ -35 & 14 & 5 \end{pmatrix}.$$

Previously, we found $\det A = -60$, and dividing the adjoint by this yields the inverse we are after:

$$A^{-1} = \begin{pmatrix} -\frac{3}{4} & \frac{3}{10} & \frac{1}{4} \\ 0 & \frac{1}{5} & 0 \\ \frac{7}{12} & -\frac{7}{30} & -\frac{1}{12} \end{pmatrix}.$$

(7) A matrix A is called *upper triangular* if all entries below the main diagonal are 0, i.e., if $A_{ij} = 0$ whenever $j < i$. For example,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 7 & 7 \\ 0 & 0 & 11 & 12 \\ 0 & 0 & 0 & 16 \end{pmatrix}$$

is upper triangular. Show that the determinant of any $n \times n$ upper triangular matrix A is the product of its entries lying on the diagonal, i.e., $\det A = A_{11}A_{22}\cdots A_{nn}$. **Hint:** Consider a row expansion along the bottom row of A. What do you observe?

Solution: The bottom row of such a matrix is of the form $\begin{pmatrix} 0 & 0 & \cdots & 0 & A_{nn} \end{pmatrix}$. Expanding along this row, we find that the determinant is $(-1)^{n+n}A_{nn} = A_{nn}$ times the determinant of the matrix with the last column and last row removed. Now this matrix is an $(n-1) \times (n-1)$ matrix, and is again upper triangular, with diagonal entries $A_{11}, \ldots A_{n-1,n-1}$. By expanding along the last row of this matrix, we find that the determinant of this matrix, for exactly the same reasons, is $A_{n-1,n-1}$ times the $(n-2) \times (n-2)$ matrix consisting of the first $(n-2)$ rows and $(n-2)$ columns of A. Continuing in this way, we eventually find that the determinant of A is equal to $A_{nn}A_{n-1,n-1}\cdots A_{22}$ times the determinant of the 1×1 matrix (A_{11}), which itself has determinant A_{11}. The claim follows.

(8) According to our definition in class, B is an inverse for A if $AB = BA = I_n$. Suppose we instead require only that $AB = I_n$. In general algebraic contexts, this will not be enough to guarantee that B is an inverse for A. However, there is enough extra structure in the theory of matrices to conclude in this situation that B is an inverse for A. This problem will guide you through the proof of this fact.

(a) Show that if A, B are square matrices of size $n \times n$ with $AB = I_n$, then A is invertible. (Hint: Use determinants).

(b) Using the notation and results of part (a), show that in fact $B = A^{-1}$ (Hint: Consider the matrix BAB).

Solution:

(a): Taking the determinant of both sides, and using the fact that a determinant of a product is the product of determinants, we have

$$\det(AB) = \det(A)\det(B) = \det(I_n) = 1.$$

Hence, we have $\det A \neq 0$, and so A is invertible.

(b): As A is invertible, there is a unique matrix A^{-1} for which $AA^{-1} = A^{-1}A = I_n$. We claim that in fact $B = A^{-1}$. The only thing we know about B is that

$$AB = I_n.$$

Multiplying both sides of this equation on the left by A^{-1} shows that $B = A^{-1}$, as desired.

(9) Suppose A is a 3×3 matrix whose third row is a linear combination of the first two rows. Show that A is not invertible and find a vector b such that $Ax = b$ has no solutions. Find a vector b for which it has infinitely many solutions.

Solution:

Suppose that the third row r_3 is a linear combination $r_3 = \alpha r_1 + \beta r_2$. Then our matrix has the shape

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \alpha a_{11} + \beta a_{21} & \alpha a_{12} + \beta a_{22} & \alpha a_{13} + \beta a_{23} \end{pmatrix}.$$

This has determinant 0, as subtracting α times the first row and β times the second row gives a row of zeros, but preserves the determinant, and hence the matrix isn't invertible. Now suppose our vector is $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3. \end{pmatrix}$. Then using row-reduction, we get a row of zeros left of the bar | by subtracting α times the first row and β times the second row from the third row:

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ \alpha a_{11} + \beta a_{21} & \alpha a_{12} + \beta a_{22} & \alpha a_{13} + \beta a_{23} & b_3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ 0 & 0 & 0 & b_3 - \alpha b_1 - \beta b_2 \end{array}\right).$$

This will have no solutions if $b_3 - \alpha b_1 - \beta b_2 \neq 0$. Otherwise, if there are no inconsistencies in the first and second rows, which we can guarantee by letting $b_1 = b_2 = 0$, then there will be at least one free variable and hence infinitely many solutions whenever $b_3 - \alpha b_1 - \beta b_2 = 0$. To find a vector with $b_3 - \alpha b_1 - \beta b_2 \neq 0$, just pick $b_1 = 0$, $b_2 = 0$, $b_3 = 1$. In the second case, just pick $b_1 = b_2 = b_3 = 0$.

In other words, we have found that $b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ yields a system with no solutions, and the vector $b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ yields a system with infinitely many solutions.

(10) Using cofactors, find the determinant and inverse of the matrix

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 5 & 1 & 4 \\ 0 & 1 & 0 \end{pmatrix}.$$

Solution:

We first compute the matrix of minors of A as

$$M = \begin{pmatrix} \det\begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix} & \det\begin{pmatrix} 5 & 4 \\ 0 & 0 \end{pmatrix} & \det\begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix} \\ \det\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} & \det\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} & \det\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \\ \det\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} & \det\begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix} & \det\begin{pmatrix} 1 & 3 \\ 5 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} -4 & 0 & 5 \\ -2 & 0 & 1 \\ 10 & -6 & -14 \end{pmatrix}.$$

By inserting signs into the minor matrix, we get the matrix of cofactors:

$$C = \begin{pmatrix} -4 & 0 & 5 \\ 2 & 0 & -1 \\ 10 & 6 & -14 \end{pmatrix}.$$

By expanding $\det A$ along the bottom row, we find that the determinant is equal to $1 \cdot C^{32} = 6$. By taking the transpose of the cofactor matrix, we get the adjugate

$$\mathrm{adj}(A) = \begin{pmatrix} -4 & 2 & 10 \\ 0 & 0 & 6 \\ 5 & -1 & -14 \end{pmatrix}.$$

Now the determinant is non-zero, and so the matrix is invertible with inverse $A^{-1} = (\det A)^{-1}\mathrm{adj}A$, which we plug in to find is

$$A^{-1} = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{5}{3} \\ 0 & 0 & 1 \\ \frac{5}{6} & -\frac{1}{6} & -\frac{7}{3} \end{pmatrix}.$$

(11) A very famous puzzle which was all the rage in the 19th century is the famous *15 puzzle*, which you have likely seen some version of. The puzzle asks you to solve the following problem. Suppose we have a 4×4 grid of sliding pieces, with 15 moving pieces and one empty square, arranged as follows:

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

That is, the numbers are all in order except that 14 and 15 are flipped, and the lower right hand corner is a blank space. Pieces can be shuffled around so that there is always one blank space; for example, one could shift the 12 down in the above configuration to give the arrangement

1	2	3	4
5	6	7	8
9	10	11	
13	15	14	12

The question is, can we shuffle around the pieces to make the puzzle pieces all line up in order, i.e., switch the 14 and the 15? The answer is no, and we can use permutations to see why. For any configuration, we can define an *invariant* as follows. Label the blank space by 16 and read define an associated permutation which assigns the sequence $1, 2, \ldots, 16$ the sequence of numbers which reads off the rows from left to right and then from top to bottom. For example, the first configuration above corresponds to the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 15 & 14 & 16 \end{pmatrix},$$

while the second configuration corresponds to

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 16 & 13 & 15 & 14 & 12 \end{pmatrix}.$$

Define the *taxicab* distance as the sum number of rows plus the number of columns that the blank space is away from the lower right corner, i.e., $0+0=0$ in the first case and $1+0=1$ in the second case. Let's say that the parity of a permutation is 0 if its even, and 1 if its odd, and similarly an integer's parity is 0 if its even and 1 if its odd. Define the *t-invariant* of a puzzle configuration as 0 if the parity of the permutation and the taxicab numbers are the same, and 1 otherwise. Note that if I make a move on the puzzle grid, I move the blank space by one, and so change the taxicab distance by exactly one, and so flip the parity of the taxi distance, and on the side of permutations, I am composing the original permutation with a transposition, and hence change the sign of the permutation as well. Thus, the t-invariant is always fixed for a given puzzle, (unless you break it into pieces with a hammer).

Now, the original 15 puzzle, where 14 and 15 are flipped, isn't solvable, as the permutation is just the transposition (14 15), which has sign -1, or as we said above parity 1, and the taxicab number is even (so has parity 0). Hence, the t-invariant is 1, while the permutation corresponding to the numbers in order is just the permutation with no transpositions (or, if you like, just write it as something like (12)(12)), and so has sign $+1$ or parity 0, while the taxicab number is still 0, and so the t-invariant is 0. These two numbers have different parities, implying the puzzle is impossible to solve.

Now, here is your question. Is the puzzle with the following configuration solvable, that is, can you shift around puzzle pieces to get the numbers 1 through 15 back in order:

3	6	4	9
7	5	2	8
12	10		1
15	14	13	11

Solution:
This puzzle corresponds to the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 3 & 6 & 4 & 9 & 7 & 5 & 2 & 8 & 12 & 10 & 16 & 1 & 15 & 14 & 13 & 11 \end{pmatrix}.$$

We find the parity of π by first writing down its cycle decomposition:

$$\pi = (1\ 3\ 4\ 9\ 12)(2\ 6\ 5\ 7)(11\ 16)(13\ 15).$$

We write each cycle as a product of transpositions to get

$$\pi = (1\ 12)(1\ 9)(1\ 4)(1\ 3)(2\ 7)(2\ 5)(2\ 6)(11\ 16)(13\ 15).$$

There are 9 transpositions here, and so the sign of π is -1 and the parity is 1. The taxicab number is $1 + 1 = 2$, which has parity 0. Thus, the t-invariant is 1. As the t-invariant is different than that t-invariant of the "solved" position with the numbers in order, this puzzle can never be solved.

(12) (Advanced) Show that if A, B are square matrices with $A + B = AB$, then $AB = BA$. (Hint: You may use that if $AB = I_n$, then $B = A^{-1}$).

Solution:

We rewrite the equation $A + B = AB$ as

$$I_n + A + B = AB + I_n,$$

or

$$I_n = AB - A - B - I_n = (A - I_n)(B - I_n).$$

Thus, using the hint, we have

$$I_n = (B - I_n)(A - I_n),$$

which upon expanding becomes

$$I_n = BA - A - B - I_n,$$

so that

$$A + B = BA.$$

But $A + B = AB$ by assumption, so $AB = BA$.