LECTURE 9: DETERMINANTS AND INVERTIBILITY, TRANSPOSES, MINORS AND COFACTORS

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1. Determinants as a criterion for detecting invertibility

We next show a few very important properties of determinants.

Theorem. *The following hold for any $n \times n$ matrices A, B.*

(1) *If A has a row of zeros, then* $\det A = 0$.
(2) $\det(AB) = \det A \cdot \det B$.
(3) *A is invertible if and only if* $\det A \neq 0$.
(4) *If A is invertible, then* $\det(A^{-1}) = (\det A)^{-1}$.

Proof. (1): If the i-th row r_i of A is zero, then $cr_i = r_i$ for any c, so that for a fixed $c \neq 0$, using mulilinearity of the determinant as a function on the rows of A, we have

$$\det A = \det(r_1, \dots, r_i, \dots, r_n) = \det(r_1, \dots, cr_i, \dots, r_n) = c \det(r_1, \dots, r_i, \dots, r_n) = c \det A,$$

which implies $\det A = 0$.

(2):

We first show it when $A = E$ is an elementary matrix. By the properties of det we showed before, $\det EB = \det B, -\det B, c \det B$, depending on whether E corresponds to adding a multiple of one row to another row, switching two rows, or multiplying a row by a constant, respectively. It is also obvious that $\det E = 1, -1, c$ accordingly, as E is obtained by performing the same elementary row operation on I_n, which has determinant 1.

In general, there are two cases. If A is invertible, by our previous theorem, A is a product of elementary matrices, so by applying the argument in the last paragraph repeatedly, the result follows. On the other hand, if A isn't invertible, then the RREF of A isn't I_n, meaning that there isn't a pivot in some row, meaning that A has a row of zeros. This means that there are elementary matrices E_j for which $E_1 \cdots E_k A$ has a row of zeros, and by the argument in the last paragraph and by part (1), the product $\det(E_1) \cdots \det(E_k) \det(A) = 0$. As the determinant of an elementary matrix is never zero, we find that $\det A = 0$. We are done if we can show $\det AB = 0$. This follows since $E_1 \cdots E_k A$ has a row of zeros, and so $E_1 \cdots E_k AB$ has a row of zeros as well.

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(3): If A is invertible, then it is a product of elementary matrices, and so by (2) has non-zero determinant. Conversely, if A isn't invertible, then we saw in the proof of (2) that $\det A = 0$.

(4): By definition, $I_n = AA^{-1}$, and so by (2), we have $\det(I_n) = 1 = \det A \det A^{-1}$, which implies the claim.

□

2. Matrix transposes

Given any matrix A of size $m \times n$, there is a matrix A^T, called the **transpose** of A, which has size $n \times m$. This is obtained by reflecting A across its main diagonal. Another way of thinking is that the rows of one are the columns of the other. Formally, we have the following.

Definition. For any matrix A of size $m \times n$, the transpose of A, written A^T, is the $n \times m$ matrix with

$$(A^T)_{ji} = A_{ij}.$$

Example. *If*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix},$$

then

$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

Example. *For any matrices A, B of the same size, and for any constant c, we have*

$$(A+B)^T = A^T + B^T,$$

$$(cA)^T = c(A^T),$$

and

$$(A^T)^T = A.$$

Example. *If $a, b \in \mathbb{R}^n$ are vectors, thought of as columns, then the dot product may be expressed*

$$a \cdot b = a^T b.$$

Transposes satisfy a property not unlike the socks and shoes property for inverses.

Lemma. *For any $m \times n$ matrix A and any $n \times \ell$ matrix B, we have $(AB)^T = B^T A^T$. More generally, we have (whenever both sides make sense):*

$$(A_1 \dots A_k)^T = A_k^T \dots A_1^T.$$

Proof. It is enough to show the first claim, as the general claim follows by repeatedly applying the general claim with $k = 2$. This is just an explicit calculation:

$$((AB)^T)_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk}B_{ki} = \sum_{k=1}^{n} B_{ki}A_{jk} = \sum_{k=1}^{n}(B^T)_{ik}(A^T)_{kj} = (B^T A^T)_{ij}.$$

□

Transposes also play nicely with determinants.

Lemma. *For any $n \times n$ matrix A,*

$$\det(A^T) = \det A.$$

Proof. There are two cases. If A is invertible, then A is a product $A = E_1 \cdots E_k$ of elementary matrices. Thus, $A^T = E_k^T \cdots E_1^T$. As a determinant of a product is the product of determinants, it is enough to show that $\det E^T = \det E$ for any elementary matrix. Indeed, if E switches two rows, or if E multiplies a row by a constant, then $E = E^T$, so their determinants are clearly equal. If E adds a multiple of one row to another, then $\det E = 1$, and E^T is another elementary matrix of the same type, so $\det(E^T) = 1$ as well.

Now, if A isn't invertible. then A^T isn't either, for if it was, then $A^T B = I_n$ implies $(A^T B)^T = B^T (A^T)^T = B^T A = I_n$, which implies that A is invertible, which is a contradiction. (If you want to be precise for the moment and check the equation $BA^T = I_n$ as well, you can, yielding that $B^T A = AB^T = I_n$, which was our original definition of invertible). Thus, the determinants of both A and A^T are zero. □

This result is very handy in many computations, as it allows us to think of columns instead of rows, which may be more convenient for explicit examples. In particular, we have the following corollary.

Theorem. *The determinant is also a multilinear, alternating function of the columns of a matrix.*

In particular, any properties you used regarding elementary row operations, hold true in exactly the same way if we replace the word "row" everywhere with "column". For example, switching two columns of a matrix multiplies the determinant by -1.

3. Minors and cofactors

Our definition of determinants is really, really, tedious to check for large matrices. The original definition requires one to evaluate n^n terms, while the Leibniz formula, which got rid of lots of terms by the alternating property, still requires one to evaluate $n!$ terms. This still grows exponentially with n. However, the determinant can be evaluated in polynomial time. So, the definitions we have given are ideal for proving theorems, but not ideal for computations. In general, somewhat like integration, finding determinants of large matrices efficiently, or finding closed formulas for determinants of infinite families of matrices is a bit of an art form. Many techniques exist, including one very handy one by Charles Dodgson (aka Lewis Carroll, of Alice in Wonderland fame).

We will now describe one method which is very widely useful, and so you never have to enumerate permutations to compute determinants again.

Definition. Given an $n \times n$ matrix A, the (i, j)-th **minor**, denoted A^{ij}, is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i-th row and the j-th column. Similarly, the (i, j)-th **cofactor** C^{ij} is defined in terms of the minor by

$$C^{ij} = (-1)^{i+j} A^{ij}.$$

Example. *For*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix},$$

we have

$$A^{23} = \det \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix} = 8 - 14 = -6.$$

We also find $C^{23} = (-1)^{2+3}(-6) = 6$.

Next time, we will see how these minors give a very simple, easily evaluated expression for determinants of matrices.