LECTURE 3: LINES AND PLANES IN THREE-DIMENSIONAL SPACE

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1. Lines in $\mathbb{R}^3$

We can now use our new concepts of dot products and cross products to describe some examples of writing down geometric objects efficiently. We begin with the question of determining equations for lines in $\mathbb{R}^3$, namely, real three-dimensional space (like the world we live in).

Aside. *Since it is inconvenient to constantly say that v is an n-dimensional vector, etc., we will start using the notation $\mathbb{R}^n$ for n-dimensional vectors. For example, $\mathbb{R}^1$ is the real-number line, and $\mathbb{R}^2$ is the xy-plane.*

Our methods will of course work in the plane, $\mathbb{R}^2$, as well. Instead of beginning with a formal statement, we motivate the general idea by beginning with an example. Every line is determined by a **point on the line** and a **direction**. For example, suppose we want to write an equation for the line passing through the point $A = (1, 3, 5)$ and which is parallel to the vector $(2, 6, 7)$. Note that if I took any non-zero multiple of v, the following argument would work equally well. Now suppose that $B = (x, y, z)$ is an arbitrary point. Then

$$B \text{ is on L} \iff \overrightarrow{AB} \text{ points in the direction of v} \iff \overrightarrow{AB} = tv, \quad t \in \mathbb{R}$$

$$\iff (x-1, y-3, z-5) = (2t, 6t, 7t) \iff \begin{cases} x = 1 + 2t \\ y = 3 + 6t \\ z = 5 + 7t \end{cases} \quad \text{for } t \in \mathbb{R}.$$

These are the **parametric equations** for L. The general formula isn't much harder to derive.

Theorem. *The line in $\mathbb{R}^3$ through the point $A = (x_0, y_0, z_0)$ in the direction of $v = (a, b, c)$ is given by the set of equations*

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct, \end{cases}$$

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where t runs over all real numbers.

Similarly, in $\mathbb{R}^2$, if $A = (x_0, y_0)$ is a point on the line and $v = (a, b)$ is a parallel vector, we arrive at the parametric system

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt, \end{cases}$$

which is equivalent after solving to t to the symmetric form

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b},$$

which is in turn equivalent to the usual equation for a line that you are familiar with.

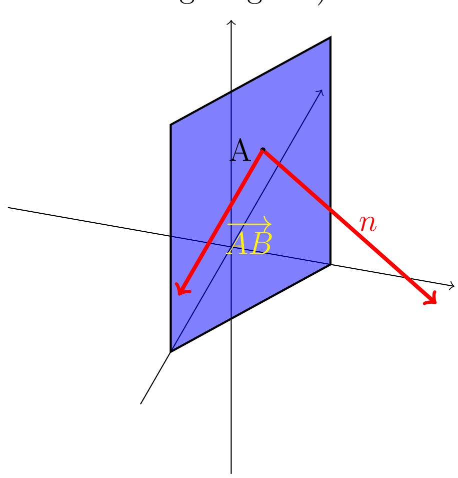
Example. *Find the line through the points $A = (2, -1, 3)$ and $B = (1, 4, -3)$.*

Solution. *We already have a point on the line (let's take A), so we need to find a vector pointing in its direction. we can take $v = \overrightarrow{AB} = (-1, 5, -6)$. This gives the system*

$$\begin{cases} x = 2 - t \\ y = -1 + 5t \\ z = 3 - 6t. \end{cases}$$

2. Planes in $\mathbb{R}^3$

We now move up one dimension. In order to find equations for a plane in $\mathbb{R}^3$, we need a **normal vector** $n = (a, b, c)$, which is orthogonal (or normal, or perpendicular) to every vector in the plane, and a point $A = (x_0, y_0, z_0)$ on the plane. In fact, the set of points on a plane is the same as the set of points whose vectors towards the fixed point A are orthogonal to n. (see the following diagram).



Thus, for a generic point $B = (x, y, z)$, we have

$$\begin{aligned} B \text{ is on the plane } &\iff \overrightarrow{AB} \perp n \iff \overrightarrow{AB} \cdot n = 0 \\ &\iff (x - x_0, y - y_0, z - z_0) \cdot (a, b, c) = 0. \end{aligned}$$

This shows the following.

Theorem. *Given a point $A = (x_0, y_0, z_0)$ on a plane and a normal vector $n = (a, b, c)$ for the plane, the equation for the plane is given by*

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Example. *Find the plane containing the three points*

$$A = (1, -2, 0), \qquad B = (3, 1, 4), \qquad C = (0, -2, 2).$$

Solution. *We already have the point $A = (1, -2, 0)$ on the plane, so we just need to find a normal vector. We first find two vectors in the plane, say*

$$\overrightarrow{AB} = (2, 3, 4), \qquad\qquad \overrightarrow{AC} = (-1, 0, 2).$$

A normal is then given by the cross product, since we saw last time that it is orthogonal to both vectors in the cross product. Thus, we can choose as a normal

$$n = \overrightarrow{AB} \times \overrightarrow{AC} = (3 \cdot 2 - 4 \cdot 0, 4 \cdot (-1) - 2 \cdot 2, 2 \cdot 0 - 3 \cdot (-1)) = (6, -8, 3).$$

Hence, we can take as an equation

$$6(x - 1) - 8(y + 2) + 3z = 0.$$

Remark. *You can easily check your answer (which is always a good idea) by plugging in the three points above into this equation.*

Example. *Find the intersection line between the two planes*

$$x - 2y + z = 6, \qquad\qquad y + z = 3.$$

Solution. *The normal vectors are $n_1 = (1, -2, 1)$ and $n_2 = (0, 1, 1)$ (read off the coefficients of x, y, z in both). The line, lying in both planes, must be orthogonal to both normals, and so parallel to the cross product*

$$n_1 \times n_2 = (-3, -1, 1).$$

To find a point on the line, we can choose any solution to the system of equations

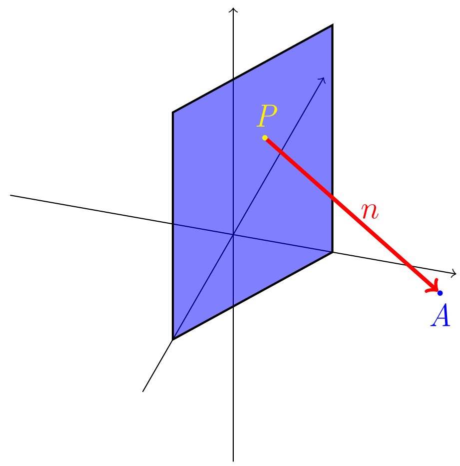
$$\begin{cases} x - 2y + z = 6 \\ y + z = 3. \end{cases}$$

We can freely choose $y = 0$, which implies $z = 3$ and $x = 3$, giving the point $A = (3, 0, 3)$. Thus, the line can be written as

$$x = 3 - 3t, \qquad y = -t, \qquad z = 3 + t.$$

Example. *Find the distance between the point $A = (1, 0, 2)$ and the plane $x - 2y + z = 9$.*

Solution. *The idea will be that in general, we first want to find line from A to the nearest point P on the plane (nearest to A, that is). As in the following generic diagram, this will be in the direction of the normal vector and passes through A.*



In our specific case, the normal vector is $n = (1, -2, 1)$, and of course passes through the point $A = (1, 0, 2)$. Thus, this line is

$$L = \begin{cases} x = 1 + t \\ y = -2t \\ z = 2 + t. \end{cases}$$

The point P lies on both L and the plane, so we solve the system of equations

$$\begin{cases} x - 2y + z = 9 \\ x = 1 + t \\ y = -2t \\ z = 2 + t \end{cases} \implies 1+t+4t+2+t = 9 \implies 6t = 6 \implies t = 1 \implies P = (2, -2, 3).$$

Thus, the distance is given by

$$\left|\overrightarrow{AP}\right| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}.$$