LECTURE 20: COMPOSITIONS OF LINEAR TRANSFORMATIONS AND CHANGE OF BASIS

MA1111: LINEAR ALGEBRA I, MICHAELMAS 2016

1. Compositions of linear transformations

In general, when we define a new mathematical object, one of the first questions we may ask is how to build new examples of that object. We have just seen some of the most basic properties of linear transformations, and how they relate to matrix multiplication. In order to use our new theory, it would be nice to be able to generate many new examples of linear transformations. Since linear transformations are just functions from a set of vectors V to a set of vectors W, in the right circumstances we can get a new function from two old ones just by taking composition of functions.

Definition. Given linear transformations $T_1 : V \to W$ and $T_2 : W \to W'$ for vector spaces V, W, W', their composition $T = T_2T_1 : V \to W'$ is their composition as functions. That is, if $v \in V$, then $T(v) = T_2(T_1(v)) \in W'$.

On the homework, you showed that this composition is indeed a linear map. We have also seen that linear transformations are related to matrices. Suppose that the dimensions of V, W, W' are n, m, p. Then after choosing ordered bases for these spaces, the associated matrix of T_1, call it A_1 has size $m \times n$, and the matrix associated to T_2, call it A_2 has size $p \times m$. Moreover, the associated matrix A to the composition has size $p \times n$. Thus, if we trace the function composition through these matrices, we find that we are taking a map determined by an $m \times n$ matrix A_1 and a $p \times m$ matrix A_2 to get a matrix A of size $n \times p$. But, the ordinary matrix product A_2A_1 is also such a matrix. You should therefore be suspicious that maybe $A_2A_1 = A$. Indeed, the following result shows that this is the case.

Theorem. *Given linear transformations $T_1 : V \to W$ and $T_2 : W \to W'$ for vector spaces V, W, W', suppose that we fix ordered bases B, C, D of V, W, W' respectively. Then we have*

$$[T_2T_1]_B^D = [T_2]_C^D \cdot [T_1]_B^C,$$

where the $\cdot$ on the right hand side is ordinary matrix multiplication.

Proof. Suppose that the dimensions of V, W, W' are n, m, p, respectively, that ordered bases are given by $B = \{v_i\}$, $C = \{w_i\}$, $D = \{w'_i\}$, and that $[T_1]_B^C = A_1$, $[T_2]_C^D = A_2$,

Date: November 24, 2016.

$[T_2T_1]_B^D = A$. Further call the components of the matrix A_1 by a_{ij} and the components of A_2 by b_{ij}. Then, denoting $T = T_2T_1$, for any basis vector $v_k \in V$ we have

$$T(v_k) = T_2(T_1(v_k)).$$

Now $T_1(v_k)$ has coordinate vector

$$A_1 \cdot [v_k]_B = A_1 \cdot e_k = \sum_{i=1}^{m} a_{ik} e_i,$$

by the definition of A_1 (note that the e_k in the second term is an element of $\mathbb{R}^n$, while the e_i in the last term is in $\mathbb{R}^m$). Applying T_2 to the corresponding image now corresponds to multiplying on the left by A_2, giving

$$A_2 \left(\sum_{i=1}^{m} a_{ik} e_i \right) = \sum_{i=1}^{m} a_{ik} A_2 e_i = \sum_{i=1}^{m} a_{ik} \sum_{j=1}^{p} b_{ji} e_j.$$

Rearranging yields

$$\sum_{j=1}^{p} \left(\sum_{i=1}^{m} b_{ji} a_{ik} \right) e_j,$$

which by definition of the matrix product is

$$\sum_{j=1}^{p} (A_2A_1)_{jk} e_j = \sum_{j=1}^{p} A_{jk} e_j = A \cdot e_k.$$

But this is just the k-th column of A. Hence, the image of v_k under the linear transformation corresponding to A_2A_1 is $T(v_k)$. Thus, the linear transformations corresponding to direct function composition and to matrix multiplication have the same effects on the basis elements v_k, and hence are the same linear transformations.

$\square$

Example. *Recall the linear map $T_\vartheta \colon \mathbb{R}^2 \to \mathbb{R}^2$ which rotates vectors be an angle $0 \leq \vartheta < 2\pi$. We saw before that the corresponding matrix for this linear transformation is*

$$A_\vartheta = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}.$$

The composition two rotations by angles ϑ and ϑ', that is, $T_{\vartheta'}T_\vartheta$ is clearly just the rotation $T_{\vartheta+\vartheta'}$. Hence, we must have

$$A_{\vartheta+\vartheta'} = A_{\vartheta'}A_\vartheta.$$

The left hand side of this equation is just

$$\begin{pmatrix} \cos(\vartheta+\vartheta') & -\sin(\vartheta+\vartheta') \\ \sin(\vartheta+\vartheta') & \cos(\vartheta+\vartheta') \end{pmatrix},$$

and the right hand side is

$$\begin{pmatrix}\cos\vartheta' & -\sin\vartheta' \\ \sin\vartheta' & \cos\vartheta'\end{pmatrix}\begin{pmatrix}\cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta\end{pmatrix} = \begin{pmatrix}\cos\vartheta\cos\vartheta' - \sin\vartheta\sin\vartheta' & -\cos\vartheta'\sin\vartheta - \cos\vartheta\sin\vartheta' \\ \cos\vartheta\sin\vartheta' + \cos\vartheta'\sin\vartheta & -\sin\vartheta\sin\vartheta' + \cos\vartheta\cos\vartheta'\end{pmatrix}.$$

Comparing corresponding entries, we find that the claim that $T_{\vartheta'}T_\vartheta = T_{\vartheta+\vartheta'}$ is equivalent to the well-known addition formulas

$$\cos(\vartheta+\vartheta') = \cos\vartheta\cos\vartheta' - \sin\vartheta\sin\vartheta',$$

$$\sin(\vartheta+\vartheta') = \cos\vartheta'\sin\vartheta + \cos\vartheta\sin\vartheta'.$$

2. Change of Basis

As we have seen, it is possible to take many possible bases of the same vector space and work with different coordinate systems for each. Our association of linear transformations to matrix multiplication was nice, but it required us to fix bases of all our vector spaces. We now consider the problem of taking a matrix representation for one basis and switching to different bases. This is called a change of basis. Such changes of variables are frequently useful; for example, you saw in your calculus class that changing variables (u-subsitution) often makes many integrals easier to compute.

To explain our main results about change of coordinate systems in his context, suppose that I have two ordered bases B, B' of a vector space V. Then I can use the most basic linear operator on V, $I_V\colon V \to V$ which acts as the identity: $I_V(v) = v$ for all $v \in V$. If I represent this matrix with the input vector space V having basis B, and the output vector space (also V) having basis B', I get the **change of basis** matrix

$$Q = [I_V]_B^{B'}.$$

If V is n-dimensional, by definition this is the $n \times n$ matrix Q whose j-th column is the coordinate vector for the j-th basis element of B in terms of the other basis B'. Note also that Q is always invertible and in fact that its inverse Q^{-1} is the change of basis matrix changing from B'-coordinates to B-coordinates.

Example. *We saw before by directly writing the corresponding systems of linear equations that if we take a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ (i.e., with respect to the standard ordered basis $E = \{e_1, \ldots, e_n\}$), then the coordinate vector $[v]_B$ with respect to a basis $B = (v_1, \ldots, v_n)$ is just $A^{-1}v$ where A is the matrix whose j-th column is v_j. This is just a special case of the above procedure, as we are changing coordinates from E to B, and $A = [I_V]_B^E$ (as the j-th column is supposed to be the j-th vector in B in the standard basis E). Thus, $A^{-1} = [I_V]_E^B$.*

Example. *Suppose $V = \mathbb{R}^2$, and consider the bases $B = \{(2,4),(3,1)\}$, $B' = \{(1,1),(1,-1)\}$. Then we find (directly or by solving the corresponding systems of linear equations) that*

$$(2,4) = 3(1,1) + (-1)\cdot(1,-1), \qquad (3,1) = 2\cdot(1,1) + 1\cdot(1,-1),$$

and hence $[(2,4)]_{B'} = (3,-1)$ *and* $[(3,1)]_B = (2,1)$. *Thus, the corresponding matrix, which changes coordinates from* B *to* B' *is the matrix with these vectors as columns, namely*

$$Q = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}.$$

For example, if we consider the vector $(5,5) \in \mathbb{R}^2$, *then in* B*-coordinates it is* $[(5,5)]_B = (1,1)$, *so that in* B'*-coordinates we have*

$$[(5,5)]_{B'} = Q \cdot [(5,5)]_B = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}.$$

Indeed, this just says that $(5,5) = 5 \cdot (1,1) + 0 \cdot (1,-1)$, *which is true.*

We can now describe what happens to the matrix representation of a linear operator $T\colon V \to V$ if we change the coordinates of V from B to B'.

Theorem. *If* $T\colon V \to V$ *is a linear operator and* B, B' *are ordered bases of* V, *then*

$$[T]_{B'}^{B'} = Q[T]_B Q^{-1},$$

where Q *is the change of basis matrix from* B *to* B'.

Proof. Let I_V be the identity transformation on V. Further suppose that $B = \{v_i\}$ and $B' = \{v_i'\}$. The j-th column of $[T]_{B'}^{B'}$ is the image $T(v_j')$ in the coordinate system B', namely

$$[T(v_j')]_{B'}.$$

Using the discussion above about change of basis matrices,

$$[T(v_j)]_{B'} = Q[T(v_j)]_B.$$

Similarly,

$$[v_j]_B = Q^{-1}[v_j]_{B'} = Q^{-1}e_j.$$

Thus,

$$[T(v_j)]_{B'} = Q[T(v_j)]_B = (QA_BQ^{-1})e_j,$$

where A_B is the matrix associated to T in B-coordinates. That is, the j-th column of the matrix $T_{B'}$ associated to T in B'-coordinates is $[T(v_j')]_{B'}$, which the last chain of equalities also shows is equal to the j-th column of QA_BQ^{-1}. Thus,

$$A_{B'} = QA_BQ^{-1},$$

where $A_{B'}$ it the matrix for T in B'-coordinates. □

Example. *Consider the linear transformation* $T\colon \mathbb{R}^2 \to' R^2$ *corresponding to the matrix*

$$A = \begin{pmatrix} 5 & -3 \\ 2 & -2 \end{pmatrix}.$$

That is, $T(v) = Av$. *Now consider the basis* $B = \{v_1, v_2\} = \{(3,1),(1,2)\}$ *of* $\mathbb{R}^2$. *The change of coordinate matrix* Q *from the standard basis* $E = \{e_1, e_2\}$ *is the inverse of the matrix whose columns are* v_1, v_2, *as we have seen above. That is,*

$$Q = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{pmatrix}.$$

Thus, the matrix representation in the coordinate system B *is*

$$QAQ^{-1} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 5 & -3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}.$$

This means that $[T(v_1)]_B = (4,0)$ *and* $[T(v_2)]_B = (0,-1)$, *or*

$$Av_1 = 4v_1, \qquad Av_2 = -v_2,$$

which is easy to check explicitly.

Finally, we remark that the above change of basis can be done for any linear transformations between (possibly different) vector spaces. The proof is similar, so we omit it, but for future reference we record it in the following result.

Theorem. *If* $T\colon V \to W$ *is a linear transformation and* A, A' *are ordered bases of* V, B, B' *ordered bases of* W, *then we can change coordinates from* A, B *to* A', B' *according to the formula:*

$$[T]_{A'}^{B'} = [I_W]_B^{B'} [T]_A^B [I_V]_{A'}^A.$$