LECTURE 19: MATRIX REPRESENTATIONS OF LINEAR TRANSFORMATIONS

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1. The matrix associated to a linear transformation

We have hinted a few times already that all linear transformations can be determined by multiplication of vectors with matrices. In this lecture, we will make this precise, and in particular we will show that once you fix bases for two finite dimensional vector spaces V and W, by taking coordinate vectors in both of these bases, we can encode any linear transformation $T\colon V \to W$ by a matrix. We will also see what the implications of ordinary matrix operations are for this fact, but the main reason you should like this is that the more abstract topic of linear transformations can be reduced to the much simpler topic of matrices. Thus, as we have seen many times in this class, basically all results in linear algebra boil down to simple things involving matrices such as row reduction and matrix multiplication.

To describe this explicitly, we require a bit more language for coordinate vectors, since there will be many different coordinate vectors with respect to different bases floating around. We first say that a set $\{v_1, \ldots, v_n\} \subset V$ is an **ordered basis** of a vector space V if it is a basis and we keep track of the ordering of the vectors $v_1, \ldots, v_n$.

Example. *The standard ordered basis of $\mathbb{R}^n$, is, of course, $\{e_1, \ldots, e_n\}$. The standard ordered basis of the polynomial space $\mathcal{P}_{\leq n}(\mathbb{R})$ is $\{1, x, \ldots, x^n\}$.*

With respect to such an ordered basis, as we have seen and computed examples of in the previous lecture, we can associate a coordinate vector to any vector $v \in V$. We will denote this vector by $[v]_B$, where B is the ordered basis of V in question.

Example. *We saw before that the linear transformation $T\colon V \to F^n$, where V is a vector space of dimension n over F, given by $T(v) = [v]_B$ for any ordered basis B of B, is an isomorphism.*

Example. *The coordinate vector of v_j with respect to an ordered basis $B = \{v_1, \ldots, v_n\}$ of V is the standard basis vector $[v_j]_B = e_j$ of $\mathbb{R}^n$.*

Now suppose that we want to describe all linear transformations from V to W, both finite-dimensional vector spaces. This will be done using the following definition, which we shall analyze shortly.

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Definition. Let V, W be finite-dimensional vector spaces, and let $T\colon V \to W$ be a linear map. Suppose that $B = \{v_1, \dots, v_n\}$ is an ordered basis for V, and that $C = \{w_1, \dots, w_m\}$ is an ordered basis for W. As $T(v_j) \in W$ for $j = 1, \dots, n$, we can write it in terms of the basis C. This allows us to define scalars a_{ij} by

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i.$$

The **matrix associated to** T is the $m \times n$ matrix A defined by

$$A_{ij} = a_{ij}.$$

Moreover, we denote this matrix by $A = [T]_B^C$.

Note that the equation

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_j$$

above implies that the j-th column $(a_{1j}, \dots, a_{mj})$ is the coordinate vector $[T(v_j)]_C$ of $T(v_j)$. This is a convenient way to think about the computation of these matrices in examples, and will underlie our description of linear transformations in terms of these associated matrices.

Example. *Consider the linear operator $T\colon \mathcal{P}_{\leq 3}(\mathbb{R}) \to \mathcal{P}_{\leq 2}(\mathbb{R})$ given by differentiation. That is, $T(f) = f'$ for any polynomial f. Let us consider the standard ordered bases of these spaces given above (call them $B = \{1, x, x^2, x^3\}$, $C = \{1, x, x^2\}$). Then to compute the associated matrix, we need to compute the image of each basis element in B. Of course, $T(1) = 0$, $T(x) = 1$, $T(x^2) = 2x$, and $T(x^3) = 3x^2$. Next, we need to expand these in terms of the basis C:*

$$\begin{aligned} T(1) &= 0 + 0 \cdot x + 0 \cdot x^2, \\ T(x) &= 1 + 0 \cdot x + 0 \cdot x^2, \\ T(x^2) &= 0 + 2x + 0 \cdot x^2, \\ T(x^3) &= 0 + 0 \cdot x + 3x^2. \end{aligned}$$

Writing using our coordinate vector notation, this says that $[T(1)]_C = (0,0,0)$, $[T(x)]_C = (1,0,0)$, $[T(x^2)]_C = (0,2,0)$, $[T(x^3)]_C = (0,0,3)$. The associated matrix is then the matrix with these vectors as its columns, namely

$$[T]_B^C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

We can use this to give a (obviously not convenient) way to differentiate polynomials. For example, the derivative of $x^3 - 7x + 2$ is given by taking the coordinate vector of this element

$$[x^3 - 7x + 2]_B = (2, -7, 0, 1),$$

taking the product

$$[T]_B^C[x^3-7x+2]_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -7 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 \\ 0 \\ 3 \end{pmatrix},$$

which is the coordinate vector in the basis C for the polynomial $-7+3x^2$, which is indeed the derivative of x^3-7x+2.

This example already illustrates exactly how the general procedure we are looking for works.

"Theorem". If V, W are finite-dimensional vector spaces with ordered bases B, C respectively, then any linear transformation $T\colon V \to W$ is encoded by (for example, can be computed on any input vector $v \in V$ using) the matrix $[T]_B^C$. In other words, linear transformations between finite-dimensional vector spaces are essentially matrices.

Proof. Assume that V is n-dimensional and W is m-dimensional We have seen before that $[T]_B^C$ defines a linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$ by matrix multiplication on the left, defined for $x \in \mathbb{R}^n$ by

$$x \mapsto [T]_B^C \cdot x.$$

We claim that T' is "essentially the same" as the original transformation T. By this, we mean that the transformation $x \mapsto [T]_B^C \cdot x$ can be used to define another linear transformation $T'\colon V \to W$. We will then show that $T = T'$, and hence that the action of T is just determined by matrix multiplication (once we take coordinates of our vectors).

To define T', we use coordinate vectors in the only ways that match up. Given a vector $v \in V$, we first take its coordinate vector $[v]_B$, which gives us a vector in $\mathbb{R}^n$. Multiplying

$$[T]_B^C \cdot [v]_B$$

gives a vector in $\mathbb{R}^m$, say $[T]_B^C \cdot [v]_B = (a_1, \ldots, a_m)$. Now we can get a vector $w \in W$ by taking the vector which has $[T]_B^C \cdot [v]_B$ as its coordinate vector (with respect to $C = (w_1, \ldots, w_m)$). Explicitly, we have

$$w = a_1w_1 + \ldots + a_mw_m.$$

We take this vector w to be the image of our function T', that is, we set

$$T'(v) = w.$$

It is not too hard to show that T' is a linear transformation, as we have seen that operations like taking coordinate vectors and matrix multiplication are all linear.

We claim that in fact $T = T'$. In the last lecture, we saw that it is enough to check this on a basis of V, say the basis $B = (v_1, \ldots v_n)$. Let us see what happens to an arbitrary vector $v_j \in B$ under T'. First, we take the coordinate vector of v_j, which is of course e_j by definition, as we saw above. We already remarked above that the definition of the matrix $[T]_B^C$ is that it is the matrix whose j-th column is $T(v_j)$. Thus, T' sends v_j to the vector in W whose coordinate vector is $[T]_B^C \cdot e_j$, which we have also seen for general matrices is the j-th column of the matrix. As required, this is also $T(v_j)$, so that

$$T(v_j) = T'(v_j),$$

for all j, and so $T(v) = T'(v)$ for all $v \in V$.

□

Example. *Consider the function $T\colon M_{2\times 2}(\mathbb{R}) \to \mathcal{P}_{\leq 2}(\mathbb{R})$ defined by*

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2.$$

This is linear, as

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) = ((a+a') + (b+b') + (2(d+d'))x + (b+b')x^2$$
$$= T\begin{pmatrix} a & b \\ c & d \end{pmatrix} + T\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

and

$$T\left(\alpha\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (\alpha a + \alpha b) + (2\alpha d)x + \alpha b x^2 = \alpha T\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let us compute its associated matrix, with respect to the bases

$$B = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

and $C = \{1, x, x^2\}$. For this, we compute the images of the basis elements in B; give them names by saying $B = \{v_1, v_2, v_3, v_4\}$. We then find that

$$T(v_1) = (1+0) + (2\cdot 0)x + 0\cdot x^2 = 1 + 0\cdot x + 0\cdot x^2,$$
$$T(v_2) = (0+1) + (2\cdot 0)x + 1\cdot x^2 = 1 + 0\cdot x + 1\cdot x^2,$$
$$T(v_3) = (0+0) + (2\cdot 0)x + 0\cdot x^2 = 0 + 0\cdot x + 0\cdot x^2,$$
$$T(v_4) = (0+0) + (2\cdot 1)x + 0\cdot x^2 = 0 + 2x + 0\cdot x^2,$$

so that

$$[T(v_1)]_C = (1,0,0), \qquad [T(v_2)]_C = (1,0,1), \qquad [T(v_3)]_C = (0,0,0), \qquad [T(v_4)]_C = (0,2,0).$$

The associated matrix is thus the matrix with these vectors as its columns, namely

$$[T]_B^C = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

To test this on an example, suppose we want to use this matrix to compute $T\left(\begin{smallmatrix} 5 & -3 \\ 0 & 1 \end{smallmatrix}\right)$*. The coordinate vector of this matrix is* $(5, -3, 0, 1)$*, and we compute*

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ -3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}.$$

That is, we have $T\left(\begin{smallmatrix} 5 & -3 \\ 0 & 1 \end{smallmatrix}\right) = 2 + 2x - 3x^2$*. Of course, this is easily double-checked to be correct by plugging into the original definition.*