LECTURE 18: INJECTIVE AND SURJECTIVE FUNCTIONS AND TRANSFORMATIONS

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1. Injective and surjective functions

There are two types of special properties of functions which are important in many different mathematical theories, and which you may have seen. The first property we require is the notion of an injective function.

Definition. A function f from a set X to a set Y is **injective** (also called one-to-one) if distinct inputs map to distinct outputs, that is, if

$$f(x_1) = f(x_2)$$

implies $x_1 = x_2$ for any $x_1, x_2 \in X$.

Example. *The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not injective as, e.g., $(-1)^2 = 1^2 = 1$. In general, you can tell if functions like this are one-to-one by using the* **horizontal line test***; if a horizontal line ever intersects the graph in two different places, the real-valued function is not injective. In this example, it is clear that the parabola can intersect a horizontal line at more than one point.*

Example. *The projection operator $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x, y) = (x, 0)$ isn't injective, as many points can project to the same point on the x-axis.*

The dual notion which we shall require is that of surjective functions.

Definition. A function $f\colon X \to Y$ is **surjective** (also called onto) if every element $y \in Y$ is in the image of f, that is, if for any $y \in Y$, there is some $x \in X$ with $f(x) = y$.

Example. *The example $f(x) = x^2$ as a function from $\mathbb{R} \to \mathbb{R}$ is also not onto, as negative numbers aren't squares of real numbers. For example, the square root of -1 isn't a real number. However, like every function, this is sujective when we change Y to be the image of the map. In this case, $f(x) = x^2$ can also be considered as a map from $\mathbb{R}$ to the set of non-negative real numbers, and it is then a surjective function. Thus, note that injectivity of functions is typically well-defined, whereas the same function can be thought of as mapping into possible many different sets Y (although we will typically use the same letter for the function anyways), and whether the function is surjective or not will depend on this choice.*

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Example. *The linear transformation which rotates vectors in* $\mathbb{R}^2$ *by a fixed angle* ϑ*, which we discussed last time, is a surjective operator from* $\mathbb{R}^2 \to \mathbb{R}^2$*. To see this, note that we can find a preimage of any vector by undoing the rotation and rotating* **clockwise** *by the same angle* ϑ*.*

Finally, we will call a function **bijective** (also called a one-to-one correspondence) if it is both injective and surjective. It is not hard to show, but a crucial fact is that functions have inverses (with respect to function composition) if and only if they are bijective.

Example. *A bijection from a finite set to itself is just a permutation. Specifically, if* X *is a finite set with* n *elements, we might as well label its elements as* $1, 2, \dots n$ *(if you take the elements of a set and paint them green, it doesn't change anything about the set), and then the permutations which we discussed before are precisely the bijections of* X*. More generally, for finite sets* X, Y *a bijection from* $X \to Y$ *exists if and only if they have the same number of elements. Surprisingly, there is even a bijection from* $\mathbb{R}$ *to* $\mathbb{R}^2$ *(although it is not an obvious one, and horribly violates usual properties of functions in calculus like differentiability). However, Cantor very famously showed that there is no bijection from the set of positive integers to* $\mathbb{R}$*, meaning that some infinities are really "larger" than other infinities. Thus, functions between sets without additional structure are too coarse to notice anything like geometry or dimension, and can be quite strange.*

In general, it can take some work to check if a function is injective or surjective by hand. However, for linear transformations of vector spaces, there are enough extra constraints to make determining these properties straightforward. Our first main result along these lines is the following.

Theorem. *A linear transformation is injective if and only if its kernel is the trivial subspace* $\{0\}$*.*

Proof. Suppose that T is injective. Then for any $v \in \ker(T)$, we have (using the fact that T is linear in the second equality)

$$T(v) = 0 = T(0),$$

and so by injectivity $v = 0$.

Conversely, suppose that $\ker(T) = \{0\}$. Then if

$$T(x) = T(y),$$

by linearity we have

$$0 = T(x) - T(y) = T(x - y),$$

so that $x - y \in \ker(T)$ But as only 0 is in the kernel, $x - y = 0$, or $x = y$. Thus, T is injective. □

Example. *This is completely false for non-linear functions. For example, the map* $f\colon \mathbb{R} \to \mathbb{R}$ *with* $f(x) = x^2$ *was seen above to not be injective, but its "kernel" is zero as* $f(x) = 0$ *implies that* $x = 0$.

Example. *Consider the linear transformation* $T\colon \mathbb{R}^2 \to \mathcal{P}_{\leq 2}$ *given by*

$$T((a,b)) = ax^2 + bx.$$

This is a linear transformation as

$$\begin{aligned} T((a,b) + (a',b')) &= T((a+a', b+b')) = (a+a')x^2 + (b+b')x \\ &= (ax^2 + bx) + (a'x^2 + b'x) = T((a,b)) + T((a',b')) \end{aligned}$$

and

$$T(c(a,b)) = T((ca,cb)) = (ca)x^2 = (cb)x = c(ax^2 + bx) = cT((a,b)).$$

Note that $\ker(T) = \{0\}$, *as if* $T((a,b)) = 0$, *then the polynomial* $ax^2 + bx$ *must be identically zero, and hence* $a = b = 0$, *so* $(a,b) = 0 \in \mathbb{R}^2$. *Thus, we automatically have that* T *is injective.*

A particularly interesting phenomenon arises when V and W have the same dimension.

Theorem. *If* V *and* W *are finite-dimensional vector spaces with the same dimension, then a linear map* $T\colon V \to W$ *is injective if and only if it is surjective. In particular,* $\ker(T) = \{0\}$ *if and only if* T *is bijective.*

Proof. By the rank-nullity theorem, the dimension of the kernel plus the dimension of the image is the common dimension of V and W, say n. By the last result, T is injective if and only if the kernel is $\{0\}$, that is, if and only if it the nullity is zero. By the rank-nullity theorem, this happens if and only if the rank of T is n, that is, if and only if the image of T is an n-dimensional subspace of the (n-dimensional) vector space W. But the only full-dimensional subspace of a finite-dimensional vector space is itself, so this happens if and only if the image is all of W, namely, if T is surjective. □

In particular, we will say that a linear transformation between vector spaces V and W of the same dimension is an **isomorphism**, and that V and W are **isomorphic**, written $V \cong W$, if it is bijective. If two vector spaces are isomorphic, we can think of them as being "the same."

Example. *For any non-negative integer* n, *all vector spaces of dimension* n *over a field* F *are isomorphic to to* F^n *(and hence to each other, as it isn't too hard to show that* $\cong$ *is an* **equivalence relation***). To find an isomorphism from a vector space* V *of dimension* n *to* F, *choose some basis* $v_1, \ldots, v_n$ *of* V. *Then we have a function* $T\colon V \to F^n$ *given by taking coordinate vectors with respect to this basis, and it isn't hard to show that this map is linear. The last theorem shows that it is bijective if the kernel is zero, as* V *and* F^n *have the same dimension by assumption. To see that this is true, suppose that some coordinate vector* $c = (c_1, \ldots, c_n)$ *of some non-zero vector* v *was zero. Then* $v = c_1v_1 + \ldots + c_nv_n = 0$, *which contradicts the assumption that* $v \neq 0$.

Another key property of linear transformations is that they are determined by their values on a basis, and can, moreover, be specified to have arbitrary values on basis elements. We have seen this when we studied multilinear maps, but for completeness we record a version of this result here.

Theorem. *Suppose that V and W are vector spaces over F and that $\{v_1, \dots, v_n\}$ is a basis for V. Then for any vectors $w_1, \dots, w_n \in W$, there is a unique linear transformation $T\colon V \to W$ for which*

$$T(v_j) = w_j,$$

with $j = 1, \dots, n$. The process of building up a linear transformation by using its values on a basis is called **extending by linearity**. *In particular, if two linear transformations have the same values on a set of basis vectors for V, then they are equal on all of V.*

Example. *Suppose that we take as a basis for $\mathbb{R}^3$ the set $\{v_1, v_2, v_3\} = \{(1,1,1), (2,3,1), (4,1,5)\}$. This is really a basis as if we put them into a matrix and take the determinant, we find*

$$\det \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix} = -2 \neq 0.$$

Suppose that we define a linear function $T\colon \mathbb{R}^3 \to \mathcal{P}(\mathbb{R})$ by setting $T(v_1) = x^2 - 5$, $T(v_2) = x^7 + 1$, $T(v_3) = 11$ and extending linearly. Then the values of T on any vector $(a,b,c) \in \mathbb{R}^3$ may be found by finding the coordinate vector and pluggin in. For instance, if $v = (1,2,5)$, then we saw before that we could find the coordinate vector c of v by taking the product

$$\begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -7 & 3 & 5 \\ 2 & -1/2 & -3/2 \\ 1 & -1/2 & -1/2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 24 \\ -13/2 \\ -5/2 \end{pmatrix}.$$

Thus,

$$T(v) = T(24v_1 - 13v_2/2 - 5v_3/2) = 24T(v_1) - \frac{13}{2}T(v_2) - \frac{5}{2}T(v_3)$$
$$= 24(x^2 - 5) - \frac{13}{2}(x^7 + 1) - \frac{55}{2} = -\frac{13}{2}x^7 + 24x^2 - 154$$

Example. *Many linear transformations from $\mathbb{R}^n \to \mathbb{R}^m$ might be specified by determining their actions on the standard unit basis $e_1, \dots, e_n$ of $\mathbb{R}^n$ and extending by linearity. As we shall see, we can easily use these values find a matrix whose associated linear transformation agrees on these basis vectors, which will illuminate the role of matrices in the theory of linear transformations.*